ON THE THEOREMS OF ŠARKOVSKII AND ŠTEFAN ON CYCLES

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(Communicated by R. Daniel Mauldin)

Abstract. New proofs of the well-known theorems of Šarkovskii and Štefan on cycles of a continuous real mapping are given.

Let us fix a real interval I and a continuous function f: I → I. For any n ∈ N we denote by f^n the n-th iterate of f. A point x ∈ I is said to be a periodic point of f if x is a fixed point of f^n for some n ∈ N. If x is periodic, the smallest n ∈ N with f^n(x) = x is called the period of x. Throughout this paper, Per(f, n) will stand for the set of all periodic points of f of the period n.

Let us consider the following ordering of the set N:
3 → 5 → 7 → ... → 2·3 → 2·5 → 2·7 → ... → 2^n ·3 → 2^n ·5 → ... → 2^n → ... → 2^2 → 2 → 1.

The aim of this paper is to give new proofs of the following two theorems.

Theorem A (A. N. Šarkovskii). Let n ∈ N. If Per(f, n) ≠ 0 then Per(f, m) ≠ 0 for any m ∈ N such that n → m.

Theorem B (P. Štefan). Let n ≥ 3 be odd. Assume that Per(f, m) = 0 for any m → n. If x₀ ∈ Per(f, n) then there exists a point x ∈ {x₀, ..., f^n−1(x₀)} such that
x < f^n−2(x) < ... < f(x) < f^2(x) < ... < f^n−1(x)
or
x > f^n−2(x) > ... > f(x) > f^2(x) > ... > f^n−1(x).

Theorem A is a well-known result proved originally by A. N. Šarkovskii [5]. Another proof of Theorem A was given by P. Štefan in [6] who filled some gaps in Šarkovskii's argument. Making use of some parts of his proof (cf. [6, Lemmas (20) and (21)]) Štefan proved Theorem B, which shows how f acts on its minimal orbits of odd period. Chung-Wu Ho and Ch. Morris [4] also proved Theorems A and B by following some ideas of P. D. Straffin Jr. concerning directed graphs (cf. [7]). Another proof of Theorem A using the work of Straffin and digraphs was given by U. Burkart [3]. A sketch of a proof

Received by the editors March 24, 1988 and, in revised form, September 1, 1988.
1980 Mathematics Subject Classification (1985 Revision). Primary 26A18; Secondary 58F20.

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0002-9939/89 $1.00 + $.25 per page

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of Theorem of Šarkovskii may be found also in the paper [2]. Some comments concerning the history of the Šarkovskii's result as well as the Burkart's proof have been presented with details by Gy. Targonski in his monograph [8, Ch. 8, §2].

Up to now the greatest difficulty in proving Theorem A has been to prove it in the particular case where \( n \) is odd (Lemma 8 below). Usually it takes the largest parts of the proofs (cf. for example [5, Theorem 4 and Lemmas 5 and 6] or [6, (15), (16), (18) and Section D]). We proceed in a different, more natural way starting with a direct proof of Theorem B. Then we can simply prove Lemma 8 making use of the following three facts: Theorem B, Lemma 6 (which is actually a result of L. Block [1]), and a simple Lemma 7. Consequently we obtain another proof of Theorem B and a new, rather short and clear proof of the Theorem of Šarkovskii. Lemmas 1, 2, 3, 4, 9, and 10 are standard and may be found for example in [6]. Nevertheless we prove them (Lemma 3 in a shorter way) for the convenience of the reader. Also Lemma 6 is presented with a short argument.

**Lemma 1.** Assume that \( m, n \in \mathbb{N} \).

(i) \( x \in \text{Per}(f^n,1) \) iff \( x \in \text{Per} f \) and the order of \( x \) divides \( n \).

(ii) \( \text{Per}(f,n) \subset \text{Per}(f^m,n/k) \) where \( k \) is the highest common divisor of \( n \) and \( m \).

(iii) If \( p \) is prime then \( \text{Per}(f,p^{n+m}) = \text{Per}(f^p, p^m) \).

**Proof of Theorem B.** If \( n = 3 \), the statement is evident. Thus we can assume that \( n \geq 5 \). Let \( C = \{x_0, \ldots, f^{n-1}(x_0)\} \). Since \( n \) is odd, there exists a point \( y \in C \) such that

\[
y < f(y) < f^2(y) \quad \text{or} \quad y > f(y) > f^2(y).
\]

Assume for example the first of the above conditions. Put

\[
x = \min\{y \in C : y < f(y) < f^2(y)\} \quad \text{and} \quad y = f^{n-1}(x).
\]

Clearly \( x < f(x) < f^2(x) \) and \( x < y \). We show that \( f^2(x) < y \). Suppose on the contrary that \( y \in (x, f(x)) \cup (f(x), f^2(x)) \). Consider the case \( y \in (x, f(x)) \). Then

\[
f(y) = x < y < f(x),
\]

whence \( \text{Per}(f,1) \cap (x, y) \neq \emptyset \). Let \( a \) be its maximal point. Since

\[
f^2(a) = a < y < f(x) = f^2(y)
\]

there exists a \( z_1 \in (a, y) \) such that \( f^2(z_1) = y \). Thus

\[
f^3(z_1) = x < z_1 \quad \text{and} \quad y < f^2(x) = f^3(y)
\]

which (cf. the assumption of the theorem, Lemma 1(i), and the definition of \( a \)) is impossible. Now suppose that \( y \in (f(x), f^2(x)) \). Put \( a = \sup \text{Per}(f,1) \cap (f(x), y) \). Since

\[
f(a) = a < y < f(f(x))
\]
there is a \( z_1 \in (f(x), a) \) such that \( f(z_1) = y \). Due to the inequalities

\[
f(y) = x < z_1 < a = f(a)
\]

we can find a \( z_2 \in (a, y) \) for which \( f(z_2) = z_1 \). Therefore we have

\[
f^3(z_2) = x < z_2 \quad \text{and} \quad y < f^2(x) = f^3(y)
\]

which again leads to a contradiction. We now have that \( f^2(x) < y \).

Let us observe that, by the inequalities

\[
f(x) < y = f^n-2(f(x)), \quad f^n-2(f^2(x)) = x < f^2(x),
\]

Lemma 1(i) and the assumption of the theorem, \( \text{Per}(f, 1) \cap (f(x), f^2(x)) \neq \emptyset \). Put

\[
a_1 = \inf \text{Per}(f, 1) \cap (f(x), f^2(x))
\]

and

\[
a_2 = \sup \text{Per}(f, 1) \cap (f(x), f^2(x)).
\]

Now, we will show, that the following condition is fulfilled for any \( k \in \{1, \ldots, (n-1)/2\} \)

\[\text{(1)} \quad \text{if } f^{2k-1}(x) < \cdots < f(x) < f^2(x) < \cdots < f^{2k}(x), \text{ then}
((f^{2k-1}(x), a_1) \cup (a_2, f^{2k}(x))) \cap \text{Per}(f, 1) = \emptyset.
\]

For fix a \( k \in \{1, \ldots, (n-1)/2\} \) and suppose on the contrary that (1) does not hold. Clearly we may assume that \( k \geq 2 \). At first let \( a \) be a point of \( \text{Per}(f, 1) \cap (f^{2k-1}(x), f(x)) \). From the inequality

\[
f^{2k-2}(f(x)) = f^{2k-1}(x) < a < a_1 = f^{2k-2}(a_1)
\]

we deduce, that \( f^{2k-2}(z_1) = a \) for a \( z_1 \in (f(x), a_1) \). Hence

\[
f(x) < f^{2k}(x) = f^{2k-1}(f(x))
\]

and

\[
f^{2k-1}(z_1) = f(f^{2k-2}(z_1)) = a < z_1,
\]

which contradicts our assumptions. Now suppose that \( \text{Per}(f, 1) \cap (f^{2i}(x), f^{2i+2}(x)) \neq \emptyset \) for an \( i \in \{1, \ldots, k-1\} \). Fix an element \( a \) of this set. Since

\[
f^{2i}(a_2) = a_2 < a < f^{2i+2}(x) = f^{2i}(f^2(x)),
\]

there exists a \( z_1 \in (a_2, f^2(x)) \) such that \( f^{2i}(z_1) = a \) which, in view of the inequalities

\[
f^{n-2}(f^2(x)) = x < f^2(x) \quad \text{and} \quad z_1 < a = f^{n-2i-2}(a) = f^{n-2}(z_1),
\]

contradicts our assumptions and finishes the proof of (1).

To complete the proof of Theorem B it is enough to show inductively that, for any \( k \in \{1, \ldots, (n-1)/2\} \)

\[\text{(2)} \quad x < f^{2k-1}(x) < \cdots < f(x) < f^2(x) < \cdots < f^{2k}(x) \leq f^{n-1}(x) = y.
\]
If \( k = 1 \) then (2) follows from the definition of \( x \). Fix a \( k \in \{1, \ldots, (n-3)/2\} \) and assume (2). At first we shall show that \( f^{2k+1}(x) \leq (x, f^{2k-1}(x)) \). Suppose that \( f^{2k+1}(x) > f^{2k}(x) \). Then for an \( a \in \text{Per}(f, 1) \cap (f^{2k}(x), y), \)
\[
f^{n-3}(a) = a < a < y = f^{n-3}(f^{2}(x))
\]
whence there exists a \( z_1 \in (a_2, f^2(x)) \) such that \( f^{n-3}(z_1) = a \). Therefore
\[
f^{n-2}(f^2(x)) = x < f^2(x) \quad \text{and} \quad z_1 < a = f(a) = f(f^{n-3}(z_1)) = f^{n-2}(z_1)
\]
which leads to a contradiction and proves that \( f^{2k+1}(x) < f^{2k}(x) \). Now suppose that \( f^{2k+1}(x) \in (a_1, f^2(x)) \). From the inequalities
\[
f^{2k+1}(a_1) = a_1 < f^{2k+1}(x) < f^2(x) = f^{2k-1}(f(x))
\]
we deduce that there exists a \( z_1 \in (f(x), a_1) \) such that \( f^{2k-1}(z_1) = f^{2k+1}(x) \). Since
\[
f^{n-2}(z_1) = f^{n-2k-1}(f^{2k-1}(z_1)) = f^{n-2k-1}(f^{2k+1}(x)) = x < z_1
\]
and
\[
f(x) < y = f^{n-2}(f(x)),
\]
there is a fixed point of \( f^{n-2} \) in \( (f(x), z_1) \) which is impossible. If \( f^{2k+1}(x) \in (f^{2k-1}(x), a_1) \) then by the inequalities
\[
f^{n-2}(f^{2k+1}(x)) = f^{2k-1}(f^2(x)) < f^{2k+1}(x)
\]
and
\[
f(x) < y = f^{n-2}(f(x)),
\]
the interval with the endpoints \( f(x) \) and \( f^{2k+1}(x) \) would contain a fixed point of \( f^{n-2} \) and this contradicts condition (1). At last suppose that \( f^{2k+1}(x) < x \). Putting \( a = \min(\{f(x)\} \cup (\text{Per}(f, 1) \cap (x, f(x)))) \) we have
\[
f^2(a) < a < f^2(x),
\]
whence \( f^2(z_1) = a_1 \) for a \( z_1 \in (x, a) \). Thus
\[
f^{2k+1}(x) < x \quad \text{and} \quad f^{2k+1}(z_1) = a_1 > z_1,
\]
which is impossible. Consequently, \( f^{2k+1}(x) \in (x, f^{2k-1}(x)) \).

Now we shall show that \( f^{2k+2}(x) \in (f^{2k}(x), y) \). Clearly we may assume that \( k \leq (n-5)/2 \). At first suppose that \( y < f^{2k+2}(x) \). In view of the inequalities
\[
f^{2k+1}(a_1) = a_1 < y < f^{2k+2}(x) = f^{2k+1}(f(x))
\]
there exists a \( z_1 \in (f(x), a_1) \) such that \( f^{2k+1}(z_1) = y \). Since
\[
f^2(f(x)) < f(x) < z_1 < a_1 = f^2(a_1),
\]
$z_1 = f^2(z_2)$ for $z_2 \in (f(x), a_1)$. Thus

$$f^{2k+3}(z_1) = f^2(y) = f(x) < z_1$$

and

$$z_2 < y = f^{2k+1}(z_1) = f^{2k+3}(z_2),$$

which is false and proves that $f^{2k+2}(x) \leq y$. Suppose that $f^{2k+2}(x) < f(x)$.

Since

$$f^2(f(x)) < f(x) < a_1 = f^2(a_1),$$

there exists a $z_1 \in (f(x), a_1)$ such that $f^2(z_1) = f(x)$. Then we have

$$f^{2k+1}(f(x)) = f^{2k+2}(x) < f(x)$$

and

$$z_1 < f^{2k}(x) = f^{2k-1}(f(x)) = f^{2k+1}(z_1),$$

which is impossible. Hence $f^{2k+2}(x) > f(x)$. Now we shall prove that

$$(3) \quad (f^{2k+1}(x), f^{2k-1}(x)) \cap \text{Per}(f, 1) = \emptyset.$$ 

Indeed, suppose that this is not the case and put $a = \min \text{Per}(f, 1) \cap (f^{2k+1}(x), f^{2k-1}(x))$. Since

$$f(a) = a < f^{2k-1}(x) \leq f(x) < a_2 < f^{2k}(x) = f(f^{2k-1}(x)),$$

there exist a $z_1 \in (a, f^{2k-1}(x))$ and $z_2, z_3 \in (f^{2k+1}(x), a)$ such that $f(z_1) = a_2$, $f(z_2) = z_1$, $f(z_3) = f^{2k-1}(x)$. Then

$$f^3(z_2) = f^2(z_1) = a_2 > z_2, \quad f^3(z_3) = f^2(f^{2k-1}(x)) = f^{2k+1}(x) < z_3,$$

contrary to the definition of $a$. This completes the proof of (3). Suppose that

$$f^{2k+2}(x) \in (f(x), a_2).$$

Then, since

$$f(f^{2k+1}(x)) < a_2 < f(f(x)),$$

we can find a $z_1 \in (f^{2k+1}(x), f(x))$ with $f(z_1) = a_2$. Furthermore, by the inequalities

$$f(f^2(x)) = f^3(x) < f(x) < f^{2k+2}(x) < a_2 = f(a_2),$$

there exist a $z_2 \in (a_2, f^2(x))$ and a $z_3 \in (f(x), a_1)$ such that $f(z_2) = f^{2k+2}(x)$ and $f(z_3) = z_2$. Thus

$$z_1 < a_2 = f^{n-2k-1}(a_2) = f^{n-2k-1}(f(z_1)) = f^{n-2k}(z_1)$$

and

$$f^{n-2k}(z_3) = f^{n-2k-2}(f^2(z_3)) = f^{n-2k-2}(f^{2k+2}(x)) = x < z_3,$$

which contradicts conditions (1) and (3). At last if $f^{2k+2}(x) \in (a_2, f^{2k}(x))$ then

$$f^{n-2}(f^{2k}(x)) = f^{2k-2}(x) < f^{2k}(x).$$
and
\[ f^{2k+2}(x) < f^{2k}(x) = f^{n-2}(f^{2k+2}(x)), \]
contrary to (1). This completes the proof of Theorem B.

The proof of Theorem A consists of the following Lemmas 2–10.

**Lemma 2.** If \( \text{Per}(f, n) \neq \emptyset \) for some \( n \in \mathbb{N} \) then \( \text{Per}(f, 1) \neq \emptyset \).

**Lemma 3.** If \( \text{Per}(f, n) \neq \emptyset \) for some \( n \geq 2 \) then \( \text{Per}(f, 2) \neq \emptyset \).

**Proof.** Fix an \( x_0 \in \text{Per}(f, n) \) and put \( x = \min\{x_0, \ldots, f^{n-1}(x_0)\} \). Let \( y \in \{x_0, \ldots, f^{n-1}(x_0)\} \) and \( z \in [x, y) \) be such that \( f(y) = x \) and \( f(z) = y \). If \( z = x \) then \( x \in \text{Per}(f, 2) \). Thus we may assume that \( x < z \). Put
\[ u = \sup\{v \in [x, z]: f^2(v) = v\}. \]
Then \( u < z \). Suppose that \( f(u) = u \) and choose an \( a \in (z, y) \cap \text{Per}(f, 1) \). Since
\[ f(u) = u < a < y = f(z) \]
we have \( a = f(b) \) for some \( b \in (u, z) \). Thus \( f^2(b) > b \) and \( f^2(z) < z \) contrary to the definition of \( u \). Consequently \( u \in \text{Per}(f, 2) \).

**Lemma 4.** Let \( n \in \mathbb{N} \). If \( \text{Per}(f, 2^n) \neq \emptyset \) then \( \text{Per}(f, 2^k) \neq \emptyset \) for any \( k \in \{0, \ldots, n\} \).

**Proof.** It is enough to consider \( k \in \{2, \ldots, n\} \) only. By Lemma 1(iii) \( \text{Per}(f^{2^n}, 2^{n-k+1}) \neq \emptyset \). Using Lemma 3 and Lemma 1(iii) again we have
\[ \emptyset \neq \text{Per}(f^{2^n}, 2) = \text{Per}(f, 2^k). \]

**Lemma 5.** Let \( k \in \mathbb{N}_0 \) and let \( l \geq 3 \) be odd. If \( \text{Per}(f, 2^k \cdot l) \neq \emptyset \) then \( \text{Per}(f, 2^n) \neq \emptyset \) for any \( n \in \mathbb{N}_0 \).

**Proof.** Use Lemma 1(ii) (taking \( n = 2^k l, m = 2^{n+k} \)), Lemma 3, and then Lemmas 1(iii) and 4.

**Lemma 6.** If
\[ f^{2k}(x_0) < \cdots < f^2(x_0) < x_0 < f(x_0) < \cdots < f^{2k-1}(x_0), \quad f^{2k+1}(x_0) \leq x_0 \]
or
\[ f^{2k}(x_0) > \cdots > f^2(x_0) > x_0 > f(x_0) > \cdots > f^{2k-1}(x_0), \quad f^{2k+1}(x_0) \geq x_0 \]
for some \( x_0 \in I \) and \( k \in \mathbb{N} \) then \( \text{Per}(f, 2k+1) \neq \emptyset \).

**Proof.** Let us assume that the first of the above conditions holds and fix an \( a \in (x_0, f(x_0)) \cap \text{Per}(f, 1) \). Let \( I_0, \ldots, I_{2k+1} \) be compact intervals such that
\[ I_{2k+1} = [x_0, f^{2k-1}(x_0)], \]
\[ I_i \text{ is contained in the compact interval with the endpoints} \]
\[ f^i(x_0) \text{ and } f^{i-2}(x_0), \quad i = 2k, \ldots, 2, \]
\[ I_1 \subset [a, f(x_0)], \quad I_0 \subset [x_0, a], \]
\[ f(I_i) = I_{i+1}, \quad i = 2k, \ldots, 0. \]
Then $f^{2k+1}(I_0) = I_{2k+1} \supset I_0$ whence we can find a $c \in I_0 \cap \text{Per}(f^{2k+1}, 1)$. Thus it follows from the relations
\[f^i(I_0) \cap I_0 \subset \{x_0\}, \quad i = 2, \ldots, 2k \quad \text{and} \quad f(I_0) \cap I_0 \subset \{a\}\]
that $c \in \text{Per}(f, 2k+1)$.

**Lemma 7.** Assume that there exists an $x_0 \in I$ such that
\[f^2(x_0) < x_0 < f(x_0) \quad \text{and} \quad f^3(x_0) \leq a\]
or
\[f^2(x_0) > x_0 > f(x_0) \quad \text{and} \quad f^3(x_0) \geq a\]
where $a \in \text{Per}(f, 1)$ belongs to the interval with the endpoints $x_0$ and $f(x_0)$. Then, for any $k \in \mathbb{N}$, $\text{Per}(f, 2k) \neq \emptyset$.

**Proof.** Assume the first of the above cases. Take $x_{-1} = f(x_0)$. Since
\[f(f(x_0)) < x_0 < a = f(a),\]
there is an $x_1 \in (a, f(x_0))$ such that $f(x_1) = x_0$. By the inequalities
\[f(a) = a < x_1 < f(x_0)\]
we can find an $x_2 \in (x_0, a)$ with $f(x_2) = x_1$. Continuing this procedure inductively we obtain a sequence $(x_n : n \in \mathbb{N})$ with the properties
\[x_{2n-2} < x_{2n} < a < x_{2n-1} < x_{2n-3} \quad \text{and} \quad f(x_n) = x_{n-1}, \quad n \in \mathbb{N}.\]

Let $I_0, \ldots, I_{2k}$ be compact intervals such that
\[I_{2k} = [f^3(x_0), f(x_0)], \quad I_{2k-1} \subset [f^2(x_0), x_0],\]
$I_i$ is contained in the compact interval with the endpoints $x_{2k-i-1}$ and $x_{2k-i-3}$, $i = 2k-2, \ldots, 0$,
\[f(I_i) = I_{i+1}, \quad i = 2k-1, \ldots, 0.\]
Then $f^{2k}(I_0) = I_{2k} \supset I_0$ whence there is a $c \in I_0 \cap \text{Per}(f^{2k+1}, 1)$. Thus, due to the relations
\[f^i(I_0) \cap I_0 = I_i \cap I_0 = \emptyset, \quad i \in \{1, \ldots, 2k-1\}\setminus\{2\},\]
and (in the case $k > 1$)
\[f^2(I_0) \cap I_0 = I_2 \cap I_0 = \{x_{2k-3}\},\]
c $\in \text{Per}(f, 2k)$ which was to be proved.

**Lemma 8.** Let $n \geq 3$ be odd. If $\text{Per}(f, n) \neq \emptyset$ then $\text{Per}(f, m) \neq \emptyset$ for any $m \in \mathbb{N}$ such that $n - m$.

**Proof.** It is enough to consider the case where $\text{Per}(f, k) = \emptyset$ for any $k - n$. In view of Theorem B we may assume for example that
\[f^{n-1}(y_0) < \cdots < y_0 < f(y_0) < \cdots < f^{n-2}(y_0)\]
for a \( y_0 \in \text{Per}(f, n) \). If \( m \) is even then, taking \( x_0 = f^{n-3}(y_0) \) in Lemma 7, we get the assertion. Assume that \( m > n \) is odd. Proceeding analogously as in the proof of Lemma 7 we get the sequence \( (y_i : i = 1, \ldots, m - n) \) such that

\[
y_0 < y_2 < \cdots < y_{m-n} < y_{m-n-1} < \cdots < y_1 < f(y_0)
\]

and

\[
f(y_i) = y_{i-1}, \quad i = 1, \ldots, m - n.
\]

Now it is enough to put \( x_0 = y_{m-n} \) and use Lemma 6.

**Lemma 9.** Let \( n \geq 2 \). If \( \text{Per}(f^2, n) \neq \emptyset \) then \( \text{Per}(f, 2n) \neq \emptyset \).

**Proof.** Let \( x_0 \in \text{Per}(f^2, n) \) and let \( r = \text{card}\{x_0, \ldots, f^{2n-1}(x_0)\} \). Clearly \( r \geq n \) and \( r \leq 2n \). Thus, \( r \in \{n, 2n\} \). If \( n \) is even then \( f^n(x_0) \neq x_0 \) whence \( r = 2n \) and \( x_0 \in \text{Per}(f, 2n) \). If \( n \) is odd and \( r = n \) we may use Lemma 8.

**Lemma 10.** Let \( n \in \mathbb{N}_0 \) and let \( l \geq 3 \) be odd. If \( \text{Per}(f, 2^n l) \neq \emptyset \) then \( \text{Per}(f, 2^n k) \neq \emptyset \) for \( m = n \) and any odd \( k \geq l \) and for any \( m > n \) and odd \( k \geq 3 \).

**Proof.** If \( n = 0 \) the assertion follows from Lemma 8. Fix an \( n \in \mathbb{N} \) and assume that the lemma holds for \( n - 1 \). By Lemma 1(ii) \( \text{Per}(f^2, 2^{n-1} l) \neq \emptyset \). Hence, in view of the induction hypothesis,

\[
\text{Per}(f^2, 2^{m-1} k) \neq \emptyset, \quad m > n \text{ and } k \geq 3 \text{ or } m = n \text{ and } k \geq 1.
\]

Now use Lemma 9.

**References**