MAXIMAL ABELIAN SUBALGEBRAS
WITH SIMPLE NORMALIZER

ROBERTO LONGO

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Abstract. All infinite factors with separable predual contain a maximal Abelian
\ast subalgebra whose normalizer generates a simple subfactor

1. INTRODUCTION

The purpose of this note is to point out that every infinite factor $M$, with
separable predual, contains a maximal Abelian \ast subalgebra $A$ whose normalizer $\mathcal{N}(A)$ generates a simple subfactor of $M$.

We recall that a subfactor $N \subset M$ is simple in $M$ if $N \vee JNJ = B(L^2(M))$ where $J$ is the modular conjugation of $L^2(M)$ (the lattice symbol $\vee$ denotes
the von Neumann algebra generated). We refer to [2,3] for the properties of
simple subfactors; what we need to know here is that $M$ always contains a
simple injective subfactor.

The proof of our result closely follows an argument of Popa [4, p. 160] with
one crucial difference: we use simple injective subfactors at the place of injective
subfactors with trivial relative commutant [1].

In this way we obtain a superposition of the results in [2,4] providing the
general construction of a new kind of MASA whose properties are more stringent
than those shared by semiregular MASA's [4]. ($A$ is semiregular if $\mathcal{N}(A)'$ is
a factor; this factor has automatically trivial relative commutant in $M$ since it
contains $A$. One calls $A$ regular if $\mathcal{N}(A)' = M$.)

2. THE CONSTRUCTION

Let $\mathcal{H}$ be a separable Hilbert space. We choose a bounded metric $d$ on the
unit ball $B(\mathcal{H})_1$ on $B(\mathcal{H})$, inducing the strong topology, and a strongly dense
sequence $\{x_n\}$ in $B(\mathcal{H})_1$. If $N \subset B(\mathcal{H})$ is a von Neumann algebra we put

$$\delta(N) = \sum_{i=1}^{\infty} \frac{d(x_i, N)}{2^i}$$
where \( d(\cdot, N) \) denotes the distance from the unit ball of \( N \).

If \( N_k \) is an increasing sequence of von Neumann algebras then \( \delta(N_k) \rightarrow 0 \) iff \( \forall N_k = B(H) \).

Let now \( M \) be an infinite factor on \( H \) with a cyclic separating vector \( \Omega \) and modular conjugation \( J \). If \( A \subset M \) is an Abelian von Neumann subalgebra we put

\[
\eta(A) = d([A\Omega], A \vee M'),
\]

where \( e = [A\Omega] \in A' \) denotes the projection onto the closure of \( A\Omega \).

The following lemma is contained in \([4,5]\) and is included for convenience; the other lemmas are standard or elementary.

**Lemma 1.** Let \( A_k \) be an increasing sequence of Abelian von Neumann subalgebras with \( A = \sqrt{A_k} \).

(i) \( A \) is maximal Abelian in \( M \) iff \( \eta(A) = 0 \) i.e. \( e \in A \vee M' \);

(ii) \( \eta(A_k) \rightarrow \eta(A) \).

Hence \( A \) is a MASA of \( M \) iff \( \eta(A_k) \rightarrow 0 \).

**Proof.**

(i) If \( A \) is a MASA of \( M \), then \( A_1 \cap M = A \) namely \( A' = A \vee M' \) and \( e \in A \vee M' \). Conversely assume \( e \in A \vee M' \) and notice that the reduced von Neumann algebra \( A_e \) is maximal Abelian in \( B(eH) \), i.e. \( A_e' = A_e \), due to the cyclicity of \( \Omega \) for \( A_e \). Since \( A' \cap M \supset A \) or \( A' \supset A \vee M' \) we have

\[
A_e = A_e' \supset (A \vee M')_e \supset A_e.
\]

thus \( (A \vee M')_e = A_e \) or \( (A' \cap M)_e = A_e \) that implies \( A' \cap M = A \) because \( \Omega \) is separating for \( M \).

(ii) The sequence of projections \( e_k = [A_k\Omega] \) converges to \( e \) increasingly, hence strongly, and \( d(e_k, e) \rightarrow 0 \); we have

\[
\eta(A) - \eta(A_k) = d(e, A \vee M') - d(e_k, A_k \vee M') \leq d(e, A \vee M') - d(e_k, A_k \vee M') + d(e, e_k) \rightarrow 0
\]

where \( d(e, A_k \vee M') \rightarrow d(e, A \vee M') \) because \( A_k \vee M' \) increases to \( A \vee M' \). \( \square \)

**Lemma 2.** Let \( N \) be an infinite factor with separable predual. There exists an increasing sequence of discrete Abelian von Neumann algebras \( B_n \) such that \( B = \bigvee B_n \) is maximal Abelian in \( N \) and all the atoms of \( B_n \) are infinite (thus equivalent) projections of \( N \).

**Proof.** The statement is clear in the case of a type I factor \( F \) (consider the step function approximation of \( L^\infty[0,1] \) and regard it as a MASA of \( B(L^\infty[0,1]) \) as usual). For a general infinite factor \( N \) note that for any MASA \( B \) of \( N \) the isomorphism of the diffuse part of \( B \) (if nonzero) with \( L^\infty[0,1] \) makes possible an atomic approximation that we only need to adjust in order that
all projections be infinite. Since $N$ is isomorphic to $N \otimes F$ we may achieve this by tensoring $B$ with a MASA of $F$ as above (the tensor product of two projections is an infinite projection if one of them is infinite).

**Lemma 3.** Let $N$ be a factor and $B$ a discrete Abelian von Neumann subalgebra of $N$. If all the atoms of $B$ are equivalent, there exists a type I subfactor $G$ of $N$ such that $B$ is a MASA of $G$. If the atoms of $B$ are infinite projections, then $G' \cap N$ is an infinite factor.

**Proof.** Let $\{p_i, i \in I\}$ be the atoms of $B$; fix $i_0 \in I$ and choose a partial isometry $v_i \in N$ from $p_{i_0}$ to $p_i$, $i \in I$. Then $\{v_i v_j^*; i, j \in I\}$ is a system of matrix units for $G$. As usual $N$ is isomorphic to $N_{p_{i_0}} \otimes G$ hence $G' \cap N$ is isomorphic to $N_{p_{i_0}}$ which is an infinite factor iff $p_{i_0}$ is an infinite projection.

**Lemma 4.** Let $N_i$ be a subfactor of the factor $M_i$ ($i = 1, 2$). The subfactor $N_1 \otimes N_2$ of $M_1 \otimes M_2$ is simple iff $N_i$ is simple in $M_1$ and $N_2$ is simple in $M_2$.

**Proof.** Let $J_i$ be the modular conjugation of $L^2(M_i)$, so that $J = J_1 \otimes J_2$ is the modular conjugation of $L^2(M_1 \otimes M_2) = L^2(M_1) \otimes L^2(M_2)$. We have

$$\left( N_1 \otimes N_2 \right) \vee J (N_1 \otimes N_2) J = \left( N_1 \vee J_1 N_1 J_1 \right) \otimes \left( N_2 \vee J_2 N_2 J_2 \right)$$

that readily entails the statement.

**Theorem 5.** Let $M$ be an infinite factor with separable predual. There exists a maximal Abelian * subalgebra $A$ of $M$ whose normalizer $\mathcal{N}(A)$ generates a simple subfactor $\mathcal{N}(A)''$ of $M$.

**Proof.** We order the pairs $(A, F)$ consisting of a type I subfactor $F$ of $M$ with infinite relative commutant $F' \cap M$ and a maximal Abelian * subalgebra $A$ of $F$ in such a way that $(A, F) \subset (\hat{A}, \hat{F})$ means that $F \subset \hat{F}$ and $\hat{A} = A \vee B$ with $B$ a MASA of $F' \cap \hat{F}$ (in other words $(A, F)$ is a tensor product component of $(\hat{A}, \hat{F})$). We will construct an increasing sequence of pairs $(A_k, F_k)$ with

$$\eta(A_k) \to 0, \quad \delta(F_k \vee JF_k J) \to 0$$

that will prove the theorem because $A = \bigvee A_k$ will be a MASA of $M$ by Lemma 1 and $\mathcal{N}(A)'' \supset R$ where $R = \vee F_k$ is simple injective subfactor of $M$.

By an iterative argument it suffices to prove separately that, for any given pair $(A, F)$, there exists a pair $(\hat{A}, \hat{F}) \supset (A, F)$ such that

(a) $\eta(\hat{A}) \leq \frac{1}{2} \eta(A)$

(b) $\delta(\hat{F} \vee J\hat{F} J) \leq \frac{1}{2} \delta(F \vee JF J)$.

To prove a) we choose in the factor $F' \cap M$ an increasing sequence of discrete Abelian von Neumann subalgebras $B_n$ such that $\vee B_n$ is maximal Abelian and all the atoms of $B_n$ are infinite, thus equivalent, in $F' \cap M$ (Lemma 2).

Since $A \vee B_n$ increases to a MASA of $M$ we have $\eta(A \vee B_n) \to 0$. Let $m$ be so large that $\eta(A \vee B_m) \leq \frac{1}{2} \eta(A)$ and let $G$ be a type I subfactor of $F' \cap M$ containing $B_m$ as a MASA and notice that the relative commutant of $G$ in
$F' \cap M$ is infinite (Lemma 3). The pair $(\tilde{A}, \tilde{F})$ with $\tilde{A} = A \cup B_m$, $\tilde{F} = F \cup G$ satisfies a).

To prove b) let $R$ be a simple injective subfactor of $F' \cap M$ [2]. Because of the tensor product decomposition $M \simeq F \otimes (F' \cap M)$ the subfactor $F \cup R$ of $M$ is simple and injective (Lemma 4).

Let $\{F_n\}$ be an increasing sequence of type I subfactors of $M$, with $F = F_1$ and $F_n' \cap M$ infinite, that generates $R$. Since $\delta(F_n \cup JF_nJ) \to 0$ we may choose $m$ so large that $\delta(F_m \cup JF_mJ) \leq \frac{1}{2}\delta(F \cup JFJ)$. Choose a MASA $B$ of $F' \cap F_m$ and set $\tilde{A} = A \cup B$, $\tilde{F} = F_m'$ so that $(\tilde{A}, \tilde{F}) \supset (A, F)$ and step b) is done. □

Remarks. Let $A$ be MASA of $M$ as in the theorem:

(a) If there exists a normal conditional expectation on $\varepsilon$ of $M$ onto $A$ then $A$ is a Cartan subalgebra of $M$. In fact if $\phi$ is a faithful normal state such that its modular group $\sigma^\phi$ leaves $A$ invariant, then $\sigma^\phi$ leaves $\mathcal{N}(A)''$ invariant, therefore by Takesaki criterium there exists a normal conditional expectation of $M$ onto $\mathcal{N}(A)''$ which implies $\mathcal{N}(A)'' = M$ [2].

(b) Since $\mathcal{N}(A)$ determines the automorphisms of $M$ [2], it is possible to analyze the automorphism group of the pair as in [6]. For example denote by $\text{Aut}(M|A)/\text{Inn}(M|A)$ the group of automorphisms of $M$ leaving $A$ pointwise invariant modulo the corresponding inner automorphism subgroup; given $\alpha \in \text{Aut}(M|A)$ the map $u \in \mathcal{N}(A) \to Z_u^\alpha \equiv \alpha(u)u^*$ is an $\alpha$-cocycle with values in $A$, that induces an isomorphism of $\text{Aut}(M|A)/\text{Inn}(M|A)$ into cohomology group $H^1_{\text{ad}}(\mathcal{N}(A), A)$.

(c) The proof of the theorem shows that there exists a simple injective subfactor $R \subset M$ such that $A$ is a regular MASA of $R$. If $M$ is already approximately finite-dimensional one obtains (by a slight variation of the argument) the known result that $M$ contains a regular MASA.

REFERENCES


Dipartimento di Matematica, Università di Roma II "Tor Vergata," Roma, Italy