THE ODLYZKO CONJECTURE AND O'HARA'S UNIMODALITY PROOF

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ABSTRACT. We observe that Andrew Odlyzko's conjecture that the Maclaurin coefficients of \( 1/[(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{k-1})] \) have alternating signs is an almost immediate consequence of an identity that is implied by Kathy O'Hara's recent magnificent combinatorial proof of the unimodality of the Gaussian coefficients.

To a true combinatorialist, a combinatorial result is not properly proved until it receives a direct combinatorial proof. This is why Kathy O'Hara's long-sought-for constructive proof of the unimodality of the Gaussian polynomials ([4], [5], see also [6]) generated so much excitement in combinatorial circles. However to non-combinatorialists, a direct combinatorial proof is "just another proof." O'Hara's proof is longer than most of the dozen previous proofs, and probably would not add any insight to anyone who is not a genuine combinatorialist. Moreover, it does not seem to be generalizable at first sight. Yet it turned out to imply a deep result (KOH) to which hitherto there was no known proof of any kind.

In this note we shall prove and generalize a conjecture of Odlyzko, using O'Hara's result. Odlyzko's results imply that for \( k \) sufficiently large, the first \( k \) coefficients in

\[
\frac{1}{(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{k-1})} = \frac{(1 - q)^k}{(1 - q)(1 - q^2) \cdots (1 - q^k)}
\]

alternate in sign. He conjectured that in fact for every \( k \geq 0 \), all of the coefficients of the above series alternate in sign. We prove the sharper result

**Theorem 1.** For any integer \( k \),

\[
\frac{(1 - q)^{(k+1)/2}}{(1 - q)(1 - q^2) \cdots (1 - q^k)}
\]

has coefficients which alternate in sign.

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Note that the exponent of \((1 - q)\) is best possible, since if \([((k + 1)/2)]\) is replaced by \([((k - 1)/2)]\) then the pole \(q = 1\) has the highest order among all the poles, all of which are roots of unity, so a partial fraction expansion would yield that the coefficients are asymptotically of the same sign.

Odlyzko has informed the authors that Theorem 1 can be used to shorten the proof in [3] by at least one third.

We will prove a more general result. Recall that the Gaussian polynomials are defined for nonnegative integers \(k\) and \(n\) by

\[
(GP) \quad G(n, k) = \left[\frac{n+k}{k}\right]_q = \frac{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{n+k})}{(1-q)(1-q^2)\cdots(1-q^k)}.
\]

If \(n\) is negative, we put \(G(n, k) = 0\). We will prove:

**Theorem 2.** For nonnegative integers \(n\) and \(k\), with \(nk\) even, \(G(n, k)(1-q)^m\) has coefficients which alternate in sign, where \(m = \min\{[(k+1)/2], [(n+1)/2]\}\).

Theorem 1 follows from Theorem 2 upon taking \(n\) even and letting \(n \to \infty\).

Theorem 2 will follow from the following amazing \(q\)-binomial identity that was derived in [7], by “algebrizing” O’Hara’s main theorem ([4], [5], [6]).

\[
(KOH) \quad G(n, k) = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} G((k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j}, d_{k-i}),
\]

where

\[
n(\lambda) = \sum_{i} (i-1)\lambda_i.
\]

The sum in (KOH) is over all partitions \(\lambda = (\lambda_1, \lambda_2, \ldots)\) of \(k\). The integer \(d_i\) is the multiplicity of \(i\) in \(\lambda\), thus in frequency notation \(\lambda = 1^{d_1}2^{d_2}\cdots i^{d_i}\). In this notation,

\[
2n(\lambda) = \sum_{i=1}^{k} (D_i^2 - D_i)
\]

where

\[
D_r = \sum_{i=r}^{k} d_i.
\]

**Proof of Theorem 2.** By symmetry in \(n\) and \(k\), we may assume that \(n\) is even. We proceed by induction on \(n\) and \(k\). Theorem 2 clearly holds for \(n = 0\) and \(k = 1\).

Let

\[
F(n, k) := (1-q)^{[(k+1)/2]}G(n, k).
\]

Then (KOH) can be rewritten as

\[
(KOH') \quad F(2n, k) = \sum_{\lambda \vdash k} (1-q)^{n(\lambda)} q^{2n(\lambda)} \prod_{i=0}^{k-1} F(2(k-i)n - 2i + \sum_{j=0}^{i-1} 2(i-j)d_{k-j}, d_{k-i}).
\]
where
\[ \alpha(\lambda) := m - \sum_{i=1}^{k} [(d_i + 1)/2]. \]

Suppose we show that \( \alpha(\lambda) \geq 0 \). If \( d \neq 1^k \), then each \( F \) on the right side of (KOH') has a second argument less than \( k \). If \( \lambda = 1^k \), the first argument of \( F \) is less than \( 2n \). Thus by induction each \( F \) is alternating. Since \( (1 - q)^{\alpha(\lambda)} \) is alternating, and the power of \( q \) is even, the left side must be alternating. So it remains to verify that \( \alpha(\lambda) \geq 0 \).

First suppose that \( n \geq [(k + 1)/2] \), so \( m = [(k + 1)/2] \). Then we will show that for any partition \( \lambda \) of \( k \), we have the inequality
\[
[(k + 1)/2] - \sum_{i=1}^{k} [(d_i + 1)/2] \geq 0.
\]

It is easy to see that (*) is
\[
[(k + 1)/2] - \text{(number of parts of } \lambda + \text{ number of } i \text{ with } d_i \text{ odd})/2.
\]
This is nonnegative, since any part \( i > 1 \) of \( \lambda \) can contribute at most one \( i \) which has \( d_i \) odd.

Next suppose that \( n < [(k + 1)/2] \), so \( m = n \). First we show
\[
n + 1 - \sum_{i=1}^{k} d_i \geq 0
\]
for all partitions \( \lambda \) of \( k \) which occur in (KOH'). The key observation is that \( F \) is zero if the first argument is negative. Thus, taking the \( i = k - 1 \) term in (KOH'), we see that
\[
2n - 2(k - 1) + \sum_{j=0}^{k-2} 2(k - 1 - j)d_{k-j} \geq 0,
\]
which is equivalent to
\[
\sum_{j=2}^{k} (j-1)d_j \geq k - 1 - n,
\]
or
\[
k = \sum_{j=1}^{k} jd_j \geq k - 1 - n + \text{number of parts of } \lambda.
\]
The final inequality implies that \( \lambda \) has at most \( n + 1 \) parts, which is (**). Clearly \( \alpha(\lambda) \geq 0 \) holds unless \( \lambda \) has \( n + 1 \) distinct parts, in which case \( \alpha(\lambda) = -1 \). In this case the \( i = k - 1 \) term in (KOH') is alternating \( (G(0,1) = 1) \) without the factor of \( (1 - q) \), so it is enough to prove that \( \alpha(\lambda) + 1 \geq 0 \).

\textbf{Remarks.} To prove Theorem 1 we need only the \( n \to \infty \) case of (KOH). John Stembridge rediscovered an identity of Hall which implies this result:

\[
\left[ \begin{array}{c} n + k \\ k \end{array} \right]_q = \sum_{d \geq k} q^{2n(d)} \left[ \begin{array}{c} n + 1 \\ d_1, \ldots, d_k \end{array} \right]_q.
\]
Then George Andrews observed that (JS) is nothing but an iteration of \( q \)-Vandermonde. Subsequently John Stembridge and Jim Joichi gave bijections that prove (JS). Their proofs are closely related to [1].

If \( nk \) is odd, Theorem 2 cannot hold, because the leading term has the wrong sign. The exponent in Theorem 2 is not always best possible: \( G(11, 6)(1 - q)^2 \) alternates in sign.

Ron Evans has made the following related conjecture. He has verified it for \( a = 1 \) from Theorem 2.

**Conjecture.** Let \( n, k, a \) be nonnegative integers, with \( k > 3 \) and \( a \) odd. Let \( G(n, k, a) \) be defined by (GP), with \( qa \) replacing \( q \) in the numerator. Then the coefficients of \( G(n, k, a)(1 - q)^{\frac{k+1}{2}} \) alternate in sign if \( nk \) is even, and the coefficients of \( G(n, k, a)(1 - q)^{\frac{k+1}{2}}/(1 - q^2) \) alternate in sign if \( nk \) is odd.

Some other remarks about (KOH) can be found in [7].

**References**

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