POLYNOMIALS OF GENERATORS
OF INTEGRATED SEMIGROUPS

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Abstract. We give general sufficient conditions on \( p \) and \( A \), for \( p(A) \) to generate an exponentially bounded holomorphic \( k \)-times integrated semigroup, where \( p \) is a polynomial and \( A \) is a linear operator on a Banach space. Corollaries include the following.

1. If \( iA \) generates a strongly continuous group and \( p \) is a polynomial of even degree with positive leading coefficient, then \( -p(A) \) generates a strongly continuous holomorphic semigroup of angle \( \frac{\pi}{2} \). (2) If \( -A \) generates a strongly continuous holomorphic semigroup of angle \( \theta \) and \( p \) is an \( n \)th degree polynomial with positive leading coefficient, with \( n(\frac{\pi}{2} - \theta) < \frac{\pi}{2} \), then \( -p(A) \) generates a strongly continuous holomorphic semigroup of angle \( \frac{\pi}{2} - n(\frac{\pi}{2} - \theta) \).

3. If \( (-A) \) generates an exponentially bounded holomorphic \( k \)-times integrated semigroup of angle \( \theta \), and \( p \) and \( \theta \) are as in (2), then \( -p(A) \) generates an exponentially bounded holomorphic \( (k + 1) \)-times integrated semigroup of angle \( \frac{\pi}{2} - n(\frac{\pi}{2} - \theta) \).

1. Introduction

In this paper, we consider linear operators, \( A \), on a Banach space, whose resolvents \( (w - A)^{-1} \), are \( O(w^k) \) outside some set \( K \), and polynomials, \( p \), such that \( p(K) \) is contained in a sector, and show that \( p(A) \) generates a holomorphic \( (k + 2) \)-times integrated semigroup. When \( p(A) \) is densely defined, it generates a \( (k + 1) \)-times integrated semigroup (Theorem 4). This is applied to polynomials of generators of holomorphic \( k \)-times integrated semigroups (Theorem 9) and polynomials of generators of strongly continuous groups (Theorem 11), as described in the abstract. Even for \( k = 0 \), that is, for polynomials of generators of strongly continuous semigroups, these results are new, except for some special cases, which we will describe below.

If \( k \) is a natural number, then the strongly continuous family of bounded operators \( \{S(t)\}_{t \geq 0} \), is an exponentially bounded \( k \)-times integrated semigroup generated by \( A \) if \( S(0) = 0 \), and there exists real \( w \) such that \((w, \infty)\) is...
contained in the resolvent set of $A$, with

$$(r - A)^{-1} = r^k \int_0^\infty e^{-r t} S(t) \, dt, \quad \text{for } r > w$$

(see Arendt [1], [2], Hieber and Kellermann [8], Neubrander [10], Thieme [15]). For convenience, we will say $0$-times integrated semigroup to mean a strongly continuous semigroup. When $A$ has nonempty resolvent set, $A$ generates an exponentially bounded $k$-times integrated semigroup if and only if the abstract Cauchy problem $u'(t) = A(u(t))$, $u(0) = x$ has a unique exponentially bounded solution, for all $x$ in $D(A^{k+1})$ (see Neubrander [10], or [5]).

It is well-known that the square of a generator of a strongly continuous group generates a $C_0$ holomorphic semigroup of angle $\frac{\pi}{4}$. (This is the special case of Theorem 11, when $p(t) = t^2$. See Goldstein [7, Chapter 2, 8.7].) The bounded holomorphic semigroup analogue of Theorem 9, for the special case of $p(t) = t^n$, is in [3], where a different method of proof was used. (See Corollary 5 of this paper.) The special case of Theorem 9, when $p(t) = t^2$, appears in Goldstein [6]. In that paper, it is also shown that, if $-A$ generates a cosine function, then $-A^{2n}$ generates a $C_0$ holomorphic semigroup of angle $\frac{\pi}{2}$, for $n = 1, 2, \ldots$.

All operators are linear, on a Banach space, $X$. We will write $e^{tA}$ for the semigroup generated by $A$, $D(A)$ for the domain of $A$. Basic material on strongly continuous semigroups may be found in Goldstein [7], Pazy [14], or van Casteren [16]. When $p$ is a polynomial, $p(A)$ is defined in the obvious way: if $p(t) = \sum_{k=0}^n a_k t^k$, then $p(A) = \sum_{k=0}^n a_k A^k$, with $D(p(A)) = D(A^n)$. All polynomials are complex valued.

2. **Main results**

**Definition 1.** $S_\theta = \{re^{i\phi} | r \geq 0, |\phi| < \theta\}$.

**Definition 2.** A $C_0$ (strongly continuous) semigroup is a $C_0$ holomorphic semigroup of angle $\theta$ ($0 < \theta \leq \frac{\pi}{4}$) if it extends to a semigroup holomorphic in the interior of $S_\theta$, and continuous on $\overline{S_\psi}$, whenever $\psi < \theta$.

A $C_0$ holomorphic semigroup of angle $\theta$ is a bounded holomorphic semigroup (BHS) of angle $\theta$ if it is bounded on $S_\psi$, whenever $\psi < \theta$.

**Remark.** It is well-known (see any of the references for $C_0$ semigroups) that $-A$ generates a BHS of angle $\theta$ if and only if $D(A)$ is dense, the spectrum of $A$ is contained in $\overline{S}_{\pi/2-\theta}$ and for all $\psi > (\frac{\pi}{2} - \theta)$, $\{||w(w - A)^{-1}||w \not\in \overline{S_\psi}\}$ is bounded.

**Definition 3.** Suppose $\frac{\pi}{2} \geq \theta > 0$. The $k$-times integrated semigroup $S(t)$ is an exponentially bounded holomorphic $k$-times integrated semigroup of angle $\theta$ if it extends to a family of operators \(\{S(z)\}_{z \in S_\theta}\) satisfying

1. The map $z \to S(z)$, from $S_\theta$ into $B(X)$, is holomorphic.
2. $d^k/dz^k S(z)$ is a semigroup.
(3) For all $\psi < \theta$, $S(z)$ is strongly continuous on $\overline{S}_\psi$.

(4) For all $\psi < \theta$, there exist finite $M_\psi$, $w_\psi$ such that $\|S(z)\| \leq M_\psi e^{w_\psi|z|}$, for all $z$ in $S_\psi$.

**Remark.** Essentially the same idea, as appears in Definition 3, when $D(A)$ is dense, is in Okazawa [13], where semigroups of class $(H_n)$ are defined.

**Remark.** In [4], we show that, if $D(A)$ is dense, then $-A$ generates an exponentially bounded holomorphic $k$-times integrated semigroup of angle $\theta > 0$ if and only if for all $\psi > (\frac{\pi}{2} - \theta)$, there exists real $c_\psi$ such that the spectrum of $A$ is contained in $c_\psi + S_\psi$, with $\{\|w^{1-k}(w-A)^{-1}\||w \notin c_\psi + S_\psi\}$ bounded. When $D(A)$ is not dense, this condition is sufficient to guarantee that $-A$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\theta$.

**Theorem 4.** Suppose $K$ is a subset of the complex plane containing the spectrum of $B$, $k$ is a nonnegative integer, $\frac{\pi}{2} > \theta > 0$, $\{\|w^{1-k}(w-A)^{-1}\||w \notin K\}$ is bounded, and $q$ is a polynomial such that $q(K)$ is contained in $S_{\theta}$. Then

(a) $-q(A)$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $(\frac{\pi}{2} - \theta)$.

(b) If $D(q(A))$ is dense, then $-q(A)$ generates an exponentially bounded holomorphic $k$-times integrated semigroup of angle $(\frac{\pi}{2} - \theta)$.

(c) If $k = 0$, $D(q(A))$ is dense, and $q(0) = 0$, then $-q(A)$ generates a BHS of angle $(\frac{\pi}{2} - \theta)$.

**Proof.** There exists finite $M$ such that

$$\|(w-A)^{-1}\| \leq M|w|^{k-1}, \quad \forall w \notin K.$$

Let $p(t) = q(t) - q(0)$, $V = S_{\theta} - q(0)$. To prove the theorem, it is sufficient to show that the spectrum of $p(A)$ is contained in $V$, with $\{\|z^{1-k}(z-p(A))^{-1}\||z \notin V\}$ bounded (see Remarks after Definitions 2 and 3).

Suppose $z$ is not in $V$. Let $\{w_j\}_{j=1}^N$ be the (not necessarily distinct) zeroes of $z - p(w)$, that is,

$$z - p(w) = \prod_{j=1}^N (w_j - w), \quad \forall \text{complex } w.$$

We have

$$z - p(A) = \prod_{j=1}^N (w_j - A).$$

For any $j$, since $p(w_j) = z$ is not in $V$, $w_j$ is not in $K$. Thus $(w_j - A)$ is invertible, and $\|(w_j - A)^{-1}\| \leq M|w_j|^{k-1}$. Thus, $z - p(A)$ is invertible, and
we obtain the following upper bound for \((z - p(A))^{-1}\).

\[
\| (z - p(A))^{-1} \| \leq \prod_{j=1}^{N} \| (w_j - A)^{-1} \| \leq M^N \prod_{j=1}^{N} |w_j|^{k-1} = M^N |z|^{k-1}.
\]

Thus, \(\{ \| z^{1-k} (z - p(A))^{-1} \| \ z \notin V \} \) is bounded, proving the theorem. \(\square\)

As an immediate corollary, we get the results of [3].

**Corollary 5.** Suppose \(-A\) generates a BHS of angle \(\theta\), and \(n(\frac{\theta}{2} - \theta) < \frac{\pi}{2}\). Then \(-A^n\) generates a BHS of angle \(\frac{\pi}{2} - n(\frac{\theta}{2} - \theta)\).

**Proof.** Suppose \(\frac{\pi}{2} > \psi > n(\frac{\theta}{2} - \theta)\). Then, since \(-A\) generates a BHS of angle \(\theta\), the spectrum of \(A\) is contained in \(S_{\psi/n}\), and \(\{ \| z^{1-k} (z - A)^{-1} \| \ z \notin S_{\psi/n} \} \) is bounded.

Let \(q(t) = t^n\). Since \(A\) generates a BHS, \(D(q(A))\) is dense. Also \(q(S_{\psi/n})\) is contained in \(S_{\psi}\), so by Theorem 4(c), \(-A^n = -q(A)\) generates a BHS of angle \(\frac{\pi}{2} - \psi\), whenever \(\frac{\pi}{2} > \psi > n(\frac{\theta}{2} - \theta)\). This implies that \(-A^n\) generates a BHS of angle \(\frac{\pi}{2} - n(\frac{\theta}{2} - \theta)\). \(\square\)

In order to apply Theorem 4 to more general polynomials of other generators, we need some elementary lemmas.

**Lemma 6.** Suppose \(E\) is a subset of the complex plane, and \(\theta \geq 0\). Then

\[
\limsup_{R \to \infty} \sup \{ |\arg(z)| |z \in E, |z| = R \} \leq \theta
\]

if and only if for all \(\psi > \theta\), there exists real \(c_\psi\) such that \(E\) is contained in \(c_\psi + S_\psi\).

**Proof.** Suppose the \(\limsup\) inequality holds, and \(\psi > \theta\). There exists finite \(M\) such that \(|\arg(z)| < \psi\), when \(z\) is in \(E\) and \(|z| \geq M\). Thus, \(E\) is contained in \(S_\psi \cup \{ z \in \mathbb{C} | |z| \leq M \}\), which may be shown to be contained in \(-M(1 + \cot \psi) + S_\psi\).

Conversely, suppose that, for all \(\psi > \theta\), there exists real \(c_\psi\) such that \(E\) is contained in \(c_\psi + S_\psi\). For any \(\psi \leq \pi\), it is not hard to see that \(\limsup_{R \to \infty} \sup \{ |\arg(z)| |z \in c_\psi + S_\psi, |z| = R \} \) equals \(\psi\). Thus

\[
\limsup_{R \to \infty} \sup \{ |\arg(z)| |z \in E, |z| = R \} \leq \psi,
\]

for all \(\psi > \theta\), which concludes the proof. \(\square\)

**Lemma 7.** If \(p(t) = t^n + q(t)\), where \(q\) is a polynomial of degree less than \(n\), \(\theta \geq 0\) and \(c\) is real, then, for all \(\psi > n\theta\), there exists real \(c_\psi\) such that \(p(c + S_\theta)\) is contained in \(c_\psi + S_\psi\).
Proof. \( \lim_{|z| \to \infty} p(c + z)/z^n = 1 \). Thus,

\[
\limsup_{R \to \infty} \sup\{ |\arg(z)||z \in p(c + S_\theta), |z| = R \}
= \limsup_{R \to \infty} \sup\{ |\arg(p(c + z))||z \in S_\theta, |z| = R \}
= \limsup_{R \to \infty} \sup\{ |\arg(z^n)||z \in S_\theta, |z| = R \}
= n\theta.
\]

Applying Lemma 6 now gives the result. \( \square \)

**Lemma 8.** Suppose \( K \) equals \( -c + S_\theta \cup c - S_\theta \), where \( c \) and \( \theta \) are nonnegative, and \( p(t) = t^{2n} + q(t) \), where \( q \) is a polynomial of degree less than \( 2n \). Then, for all \( \psi > 2n\theta \), there exists real \( c \) such that \( p(K) \) is contained in \( c + S_\psi \).

Proof. Let \( K^+ = -c + S_\theta \). Since \( K = K^+ \cup -K^+ \), it is sufficient, by Lemma 6, to show that \( \limsup_{R \to \infty} \sup\{ |\arg(p(z))||z \in \pm K^+, |z| = R \} = 2n \); this follows exactly as in the proof of Lemma 7. \( \square \)

**Theorem 9.** Suppose \( -A \) generates an exponentially bounded holomorphic \( k \)-times integrated semigroup of angle \( \theta \), \( p(t) = t^n + q(t) \), where \( q \) is a polynomial of degree less than \( n \), and \( n(\frac{n}{2} - \theta) < \frac{n}{2} \). Then

(a) \(-p(A)\) generates an exponentially bounded holomorphic \((k + 1)\)-times integrated semigroup of angle \( \frac{n}{2} - n(\frac{n}{2} - \theta) \).
(b) If \( D(p(A)) \) is dense, then \(-p(A)\) generates an exponentially bounded holomorphic \( k \)-times integrated semigroup of angle \( \frac{n}{2} - n(\frac{n}{2} - \theta) \).
(c) If \( k = 0 \), then \(-p(A)\) generates a \( C_0 \) holomorphic semigroup of angle \( \frac{n}{2} - n(\frac{n}{2} - \theta) \).

Proof. Suppose \( \frac{n}{2} > \psi > n(\frac{n}{2} - \theta) \). Choose \( \phi \) such that \( \frac{n}{2} > \phi > \frac{n}{2} - \theta \). Since \(-A\) generates an exponentially bounded holomorphic \( k \)-times integrated semigroup of angle \( \theta \), there exists real \( c \) such that the spectrum of \( A \) is contained in \( c + S_\phi \), and \( \{ \|w^{1-k}(w - A)^{-1}\||w \notin c + S_\phi \} \) is bounded.

By Lemma 7, there exists real \( c_\psi \) such that \( p(c + S_\phi) \) is contained in \( c_\psi + S_\psi \).

By Theorem 4(a), \( c_\psi I - p(A) \) generates an exponentially bounded holomorphic \((k + 1)\)-times integrated semigroup of angle \( \frac{n}{2} - \psi \).

Thus, whenever \( \frac{n}{2} > \psi > n(\frac{n}{2} - \theta) \), \(-p(A)\) generates an exponentially bounded holomorphic \((k + 1)\)-times integrated semigroup of angle \( \frac{n}{2} - \psi \). This implies (a).

The same argument, using Theorem 4(b), implies (b).

For (c), note that, since \(-A\) generates a \( C_0 \) semigroup, \( D(p(A)) \) is dense. Thus the argument above, with Theorem 4(c), implies (c). \( \square \)

**Corollary 10.** Suppose \( p \) is a polynomial with positive leading coefficient.

(a) If \(-A\) generates a \( C_0 \) holomorphic semigroup of angle \( \frac{n}{2} \), then \(-p(A)\) generates a \( C_0 \) holomorphic semigroup of angle \( \frac{n}{2} \).
(b) If \(-A\) generates an exponentially bounded holomorphic \(k\)-times integrated semigroup of angle \(\frac{\pi}{2}\), then \(-p(A)\) generates an exponentially bounded holomorphic \((k+1)\)-times integrated semigroup of angle \(\frac{\pi}{2}\).

The following theorem would follow from Theorem 9 and the fact that the square of the generator of a \(C_0\) group generates a \(C_0\) holomorphic semigroup, if \(q(t)\) contained only even powers of \(t\). Theorem 11 is more general, in that \(q\) may be any polynomial of degree less than \(2n\).

**Theorem 11.** Suppose \(iA\) generates a strongly continuous group, and \(p(t) = t^{2n} + q(t)\), where \(q\) is a polynomial of degree less than \(2n\). Then \(-p(A)\) generates a \(C_0\) holomorphic semigroup of angle \(\frac{\pi}{2}\).

**Proof.** Suppose \(\frac{\pi}{2} > \phi > 0\). Since \(iA\) generates a \(C_0\) group, there exists positive \(r\) such that the spectrum of \(A\) is contained in the horizontal strip \(\{z \in \mathbb{C}||\text{Im}(z)| < r\}\), with \(\{||\text{Im}(z)(z - A)^{-1}||\text{Im}(z)\} \geq r\} \) bounded.

Let \(c = r \cot(\frac{\phi}{4n})\), \(K = (c + S_{\phi/4n}) \cup (c - S_{\phi/4n})\).

Since \(\{z \in \mathbb{C}||\text{Im}(z)| < r\} \) is contained in \(K\), and \(\{|z/\text{Im}(z)||z \notin K\}\) is bounded, it follows that \(\{||z(z - A)^{-1}||\text{Im}(z)|z \notin K\}\) is bounded.

By Lemma 8, there exists real \(c_\phi\) such that \(p(K)\) is contained in \(c_\phi + S_\phi\).

Since \(iA\) generates a \(C_0\) group, \(D(p(A))\) is dense. By Theorem 4(c), \(c_\phi I - p(A)\) generates a BHS of angle \(\frac{\pi}{2} - \phi\). Thus, for any positive \(\phi\), \(-p(A)\) generates a \(C_0\) holomorphic semigroup of angle \(\frac{\pi}{2} - \phi\), so that \(-p(A)\) generates a \(C_0\) holomorphic semigroup of angle \(\frac{\pi}{2}\). \(\Box\)

The same argument, using Theorem 4(a) and (b), instead of (c), gives the following.

**Theorem 12.** Suppose both \(iA\) and \(-iA\) generate exponentially bounded \(k\)-times integrated semigroups and \(p\) is as in Theorem 11. Then

(a) \(-p(A)\) generates an exponentially bounded holomorphic \((k+1)\)-times integrated semigroup of angle \(\frac{\pi}{2}\).

(b) If \(D(p(A))\) is dense, then \(-p(A)\) generates an exponentially bounded holomorphic \(k\)-times integrated semigroup of angle \(\frac{\pi}{2}\).

**Corollary 13.** Suppose \(p\) is an arbitrary polynomial.

(a) If both \(iA\) and \(-iA\) generate exponentially bounded \(k\)-times integrated semigroups, then \(-|p|^2(A)\) generates an exponentially bounded holomorphic \((k+1)\)-times integrated semigroup of angle \(\frac{\pi}{2}\).
(b) If, in addition to (a), \( D(|p|^2(A)) \) is dense, then \(-|p|^2(A)\) generates an exponentially bounded holomorphic \(k\)-times integrated semigroup of angle \(\frac{\pi}{2}\).

(c) If \(iA\) generates a \(C_0\) group, then \(-|p|^2(A)\) generates a \(C_0\) holomorphic semigroup of angle \(\frac{\pi}{2}\).

3. **Examples**

The most obvious application of Theorem 11 is to choose \(A = i\frac{d}{dx}\), on \(L^p(\mathbb{R})\), for \(1 \leq p < \infty\). For \(p = \infty\), the resolvent of \(iA\) still satisfies the same growth conditions as the generator of a \(C_0\) group, thus the proof of Theorem 11 with Theorem 4(a) replacing Theorem 4(c), gives us (b) of the following.

**Example 1.** Let \(A = i\frac{d}{dx}\), on \(L^p(\mathbb{R})\) \((1 \leq p \leq \infty)\), with maximal domain, and \(B = (-1)^n\left(i\frac{d}{dx}\right)^{2n} + q(i\frac{d}{dx}) = A^{2n} + q(A)\), where \(q\) is a polynomial of degree less than \(2n\).

(a) If \(1 \leq p < \infty\), then \(B\) generates a \(C_0\) holomorphic semigroup of angle \(\frac{\pi}{2}\).

(b) If \(p = \infty\), then \(B\) generates an exponentially bounded holomorphic once-integrated semigroup of angle \(\frac{\pi}{2}\).

**Remark.** In Hieber and Kellermann [8], it is shown that \(Q(i\frac{d}{dx})\), on \(L^p(\mathbb{R})\), \(1 \leq p \leq \infty\), generates an exponentially bounded once-integrated semigroup, whenever \(Q(\mathbb{R})\) is contained in a left half-plane.

Suppose \(B\) generates an exponentially bounded \(k\)-times integrated semigroup. In Neubrander [11], and [5], it is shown that

\[
A 
= \begin{bmatrix} B & B \\ 0 & B \end{bmatrix}, \quad D(A) = D(B) \times D(B),
\]

(1)

generates an exponentially bounded \((k + 1)\)-times integrated semigroup. It is straightforward to show that this integrated semigroup is holomorphic if the integrated semigroup generated by \(B\) is (see Neubrander and deLaubenfels [12]). Thus we have the following, using the fact that

\[
A^n = \begin{bmatrix} B^n & nB^n \\ 0 & B^n \end{bmatrix},
\]

for all \(n\).

**Example 2.** (a) Suppose both \(iB\) and \(-iB\) generate exponentially bounded \(k\)-times integrated semigroups, and \(p(t) = t^{2n} + q(t)\), where \(\text{deg}(q) < 2n\). Then

\[
\begin{bmatrix} p(B) & Bp'(B) \\ 0 & p(B) \end{bmatrix}
\]

(2)

generates a holomorphic exponentially bounded \((k + 2)\)-times integrated semigroup of angle \(\frac{\pi}{2}\). If \(D(B^{2n})\) is dense, then it generates a holomorphic exponentially bounded \((k + 1)\)-times integrated semigroup.
(b) Suppose $-B$ generates an exponentially bounded holomorphic $k$-times integrated semigroup of angle $\theta$, and $p$ is an $n$th degree polynomial with positive leading coefficient, with $n((\frac{k}{2} - \theta) < \frac{k}{2}$. Then the operator in (3.2) generates an exponentially bounded holomorphic $(k+2)$-times integrated semigroup of angle $\frac{k}{2} - n((\frac{k}{2} - \theta)$.

If $D(B^n)$ is dense, then it generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\frac{k}{2} - n((\frac{k}{2} - \theta)$.

In Example 2(a), $B$ could be $Q(i\frac{d}{dx})$, on $L^p(\mathbb{R})(1 \leq p \leq \infty)$, when $Q(\mathbb{R})$ is contained in a left half-plane (see Remark after Example 1).

References


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