THE ASYMPTOTICS OF THE DETERMINANT FUNCTION FOR A CLASS OF OPERATORS

LEONID FRIEDLANDER

(Communicated by Paul S. Muhly)

Abstract. Let $A$ be an elliptic pseudodifferential operator on a closed manifold $M$ and $\ord A > \dim M$. We derive the asymptotics of $\log \det(1 + eA^{-1})$ when $e \to \infty$. The constant term of this asymptotics equals $-\log \det A$.

Introduction

Let $A$ be an elliptic pseudodifferential operator on a closed manifold $M$ and $n = \ord A > \dim M = d$. Suppose that $\arg(A\varphi, \varphi) \leq \alpha$, $0 \leq \alpha < \pi/2$. One can define complex powers of the $A$ and $A^{-s}$ are operators from the trace class when $\Re s > d/n$. The $\zeta$-function $\zeta(s) = \Tr A^{-s}$ is holomorphic in the half-plane $\Re s > d/n$ and admits analytic continuation to a meromorphic function in the whole complex plane. The poles of this function are located in the points $(d - j)/n$; $j = 0, 1, 2, \ldots$. The point 0 is not really a pole of $\zeta(s)$. If the operator $A$ is differential or it is a power of a differential operator some other poles drop. The residues of $\zeta(s)$ and the numbers $\zeta(0)$, $\zeta(-1)$, $\ldots$ can be calculated if one knows the complete symbol of the operator $A$ [1–3]. Particularly the point $s = 0$ is a regular point of the $\zeta$-function and one can define

$$W(A) = \log \det A = -\zeta'(0).$$

It is easy to check that this definition gives us the generalization of a finite dimensional operator's determinant.

On the other hand the inverse operator $A^{-1}$ belongs to the trace class and the determinants

$$D(e) = \det(1 + eA^{-1}) = \prod_{j=1}^{\infty}(1 + e\lambda_j) \quad (\lambda_j = \mu_j^{-1})$$

Received by the editors November 14, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 47G05; Secondary 47B25, 47B38.

©1989 American Mathematical Society

0002-9939/89 $1.00 + .25$ per page
are defined. The infinite product in the right hand side of (2) converges. It is more convenient to consider the function

\[ d(\varepsilon) = \log D(\varepsilon) = \sum_{j=1}^{\infty} \log(1 + \varepsilon \lambda_j). \]

We are going to investigate the asymptotic expansion of the \( d(\varepsilon) \) when \( \varepsilon \to +\infty \). In particular the determinant \( W(A) \) will appear in this expansion.

We think there are two reasons why this expansion is interesting. The first reason is the connection between the Fredholm determinant \( \det(1 + \varepsilon A^{-1}) \) and the determinant of the elliptic operator \( A \) we shall obtain from this expansion.

The second reason is that the value \( \det(1 + \varepsilon A^{-1}) \) appears in the measure theory as the result of integration of \( \exp(-\varepsilon (x, x)/2) \) over a Hilbert space with respect to a Gaussian measure with average 0 and correlation operator \( A^{-1} \) (e.g. see [4]).

The fact that \( \det(1 + \varepsilon A^{-1}) \) admits asymptotic expansion for big \( \varepsilon \) is not surprising. Let us differentiate formally the equality (3) with respect to \( \varepsilon \):

\[ d'(\varepsilon) = \sum_{j=1}^{\infty} \lambda_j (1 + \varepsilon \lambda_j)^{-1} = \sum_{j=1}^{\infty} (\mu_j + \varepsilon)^{-1} = \text{Tr}(A + \varepsilon)^{-1}. \]

The trace of the resolvent \( \text{Tr}(A + \varepsilon)^{-1} \) admits asymptotic expansion when \( \varepsilon \to \infty \) [5]. So one can justify the possibility of the last expansion's integration and receive the asymptotics for \( d(\varepsilon) \). The only thing we can not get in this way is the constant of integration, i.e. the term with \( \varepsilon^0 \). We shall see later that this term is of the most interest. So we shall derive the asymptotics for \( d(\varepsilon) \) directly.

The aim of this paper is to prove

**Theorem.** The function \( d(\varepsilon) \) admits asymptotic expansion

\[ d(\varepsilon) \sim \sum_{k=-d}^{\infty} p_k \varepsilon^{-k/n} + \sum_{j=0}^{\infty} q_j \varepsilon^{-j} \log \varepsilon \]

with

\[ p_0 = -\log \det A. \]

2. **Connection between \( d(\varepsilon) \) and \( \zeta(\varepsilon) \)**

In this section we shall prove

**Proposition 1.** Let \( d/n < a < 1 \). Then

\[ d(\varepsilon) = \frac{1}{2\pi i} \int_{\text{Res}=a} \varepsilon^s b(s) \zeta(s) \, ds \]

where

\[ b(s) = \frac{1}{s} \int_{0}^{\infty} \frac{t^{-s}}{1+t} \, dt. \]
Proof. It is convenient to introduce the new argument \( \omega = \log \varepsilon \). Let \( d^*(\varepsilon) = d(e^{\omega}) \). Then formula (6) can be rewritten in the form

\[
d^*(\omega) = \frac{1}{2\pi i} \int_{\text{Res} = a} e^{\varepsilon \omega} b(s) \zeta(s) \, ds.
\]

Integrating by parts and changing the argument \( t = e^{\omega} \) we derive

\[
b(s) = \int_0^\infty t^{-s-1} \log(1 + t) \, dt = \int_{-\infty}^\infty e^{-\varepsilon \omega} \log(1 + e^{\varepsilon \omega}) \, d\omega.
\]

Note that the last formula gives the Fourier transformation \( \log(1 + e^{\varepsilon \omega}) \rightarrow b(is) \).

So by the inverse Fourier formula

\[
\log(1 + e^{\varepsilon \omega}) = \frac{1}{2\pi i} \int_{\text{Res} = a} b(s)e^{\varepsilon \omega} \, ds.
\]

After substituting \( e\lambda \) instead of \( e^{\varepsilon \omega} \) we derive

\[
\log(1 + e\lambda) = \frac{1}{2\pi i} \int_{\text{Res} = a} e^s b(s)\lambda^s \, ds.
\]

Now to obtain (6) we have to sum the last identities for \( \lambda = \lambda_j \). Note that the summation and integration operations in the right hand side commute and \( \zeta(s) \) is the sum of \( \lambda_j^s \).

3. Analytical properties of \( b(s) \)

We shall obtain asymptotic expansion for \( d^*(\omega) \) (or \( d(\omega) \)) by shifting the contour in (8) to the left. So we must have analytical continuation of \( b(s) \) into the left half-plane and we must estimate \( |b(s)| \) and \( |\zeta(s)| \) for large \( |\text{Im} s| \). In this section we shall prove

Proposition 2. The function \( b(s) \) admits analytical continuation to a meromorphic function into the half-plane \( \text{Re} s < 1 \). It has a pole of order 2 into the point \( s = 0 \) and simple poles into the points \( -1, -2, \ldots \); \( \text{Re}_{s=0} b(s) = 0 \), \( \text{Re}_{s=0} s b(s) = 1 \). The function \( b(s) \) satisfies the following estimate

\[
|b(\sigma + i\tau)| \leq C(\sigma)|\tau|^{\sigma-1} \exp\left(-\frac{\pi}{2} |\tau| \right), \quad \sigma < 1, |\tau| \geq 1.
\]

Proof. Let

\[
J_n = \int_0^\infty t^{-s}(1 + t)^n \, dt, \quad n \geq 1.
\]

Integrating by parts we obtain

\[
J_n = -\frac{n}{s-1} \int_0^\infty \frac{t^{1-s}}{(1 + t)^{n+1}} \, dt = -\frac{n}{s-1} J_n + \frac{n}{s-1} J_{n+1}.
\]

So

\[
\frac{s + n - 1}{s - 1} J_n = \frac{n}{s - 1} J_{n+1}.
\]
and
\[ J_n = \frac{n}{s + n - 1} J_{n+1}. \]

Thus
\[ J_1 = \frac{k!}{s(s + 1) \cdots (s + k - 1)} \int_0^\infty t^{-s} (1 + t)^{-k} \, dt. \]

The integral in the right hand side of (10) is absolutely convergent in the strip \(-k < \text{Re} s < 1\). So the assertion about analytical continuation of \(b(s)\) is proved.

Applying (10) with \(k = 1\) we get
\[
\text{Res}_{s=0} s \cdot b(s) = \int_0^\infty \frac{dt}{(1 + t)^2} = 1; \\
\text{Res}_{s=0} b(s) = \frac{d}{ds} \int_0^\infty \frac{t^{-s}}{(1 + t)^2} \, dt|_{s=0} = -\int_0^\infty \frac{\log t}{(1 + t)^2} \, dt.
\]

Changing the argument \(t \to 1/u\) in the last integral one can see that
\[
\int_0^\infty \frac{\log t}{(1 + t)^2} \, dt = -\int_0^\infty \frac{\log u}{(1 + u)^2} \, du,
\]
so
\[ \text{Res}_{s=0} b(s) = 0. \]

To prove (9) we shall use the representation (10). The number \(k\) in (10) will be large for large \(|\tau|\) (really \(k \sim |\tau|^2\)). Suppose that \(-\infty < \sigma_0 < \sigma < \sigma_1 < 1\) with some \(\sigma_0\) and \(\sigma_1\). All constants in the following estimates will depend on \(\sigma_0\) and \(\sigma_1\) only. We shall not numerate these constants. They will be designated by the same letter \(C\).

To begin with, let us estimate the integral (10). This integral splits into
\[
J = J_1 + J_2 = \int_0^{1/k} t^{-s} (1 + t)^{-k} \, dt + \int_{1/k}^\infty t^{-s} (1 + t)^{-k-1} \, dt.
\]

The estimation of \(J_1\) is very easy:
\[
|J_1| \leq C \int_0^{1/k} t^{-\sigma} \, dt \leq C k^{\sigma-1}.
\]

To estimate \(J_2\) let us suppose that \(0 \leq \sigma \leq 1\) (note that we shift \(s\) from the strip \(\sigma_0 < \sigma < \sigma_1\) !). The function \(t^{-\sigma}\) decreases, so
\[
|J_2| \leq k^\sigma \int_{1/k}^\infty (1 + t)^{-(k+1)} \, dt \leq k^{\sigma-1}.
\]
If $\sigma < 0$ we shall integrate by parts $n = \left\lceil \sigma \right\rceil$ times ($\lfloor x \rfloor$—entire part of the number $x$).

\[
|J_2| \leq \int_{1/k}^{\infty} t^{-\sigma} (1 + t)^{-k-1} \, dt
\]

\[
= k^{\sigma-1} \left(1 + \frac{1}{k}\right)^k - \frac{\sigma}{k} \int_{1/k}^{\infty} t^{-\sigma-1} (1 + t)^{-k} \, dt = \ldots
\]

\[
= k^{\sigma-1} \left(1 + \frac{1}{k}\right)^{-k} - \sigma \frac{k^\sigma}{k-1} \left(1 + \frac{1}{k}\right)^{-(k-1)} + \ldots
\]

\[
+ (-1)^n \frac{\sigma(\sigma+1) \cdots (\sigma+n-1)}{k(k-1) \cdots (k-n+1)} \int_{1/k}^{\infty} t^{-fr(\sigma)} (1 + t)^{-k+n-1} \, dt.
\]

Every term except the last is obviously estimated by $Ck^{\sigma-1}$. The integral in the last term is the integral of the form we have just investigated ($k \rightarrow k-n$; $\sigma \rightarrow fr(\sigma)$), so it is estimated by $k^{fr(\sigma)-1}(fr(\sigma) = \sigma - \lfloor \sigma \rfloor)$. The product which stands before the integral is of the order $k^{-n} = k^{-\lfloor \sigma \rfloor}$. Thus the last term is also estimated by $k^{\sigma-1}$. Finally,

\[
|J_2| \leq Ck^{\sigma-1}
\]

and

\[
\left| \int_0^{\infty} t^{-s} (1 + t)^{-k-1} \, dt \right| \leq Ck^{\sigma-1}.
\]

Now we intend to estimate the product which stands before the integral in the right hand side of (10). Denote

\[
q(k) = \log \left| \frac{k!}{s^s(s+1) \cdots (s+k-1)} \right|
\]

and

\[
l = \left\lceil \sigma_0 \right\rceil + 1.
\]

We have

\[
r(k) = \log k! - 2 \log |s| - \sum_{j=1}^{l-1} \log |s+j| - \sum_{j=l}^{k-1} \log |s+j|
\]

\[
\leq \log k! - (l+1) \log |s| - \frac{1}{2} \sum_{j=1}^{k-1} \log ((\sigma+j)^2 + \tau^2)
\]

(12)

\[
= \log k! - (l+1) \log |\tau| - \sum_{j=1}^{k-1} \log (\sigma+j) - \frac{1}{2} \sum_{j=l}^{k-1} \log \left(1 + \frac{\tau^2}{(\sigma+j)^2}\right)
\]

\[
= \log ! + \sum_{j=0}^{k-l-1} \log \left(1 + \frac{k-j}{k-j+\sigma-1}\right) - \frac{1}{2} \sum_{j=l}^{k-1} \log \left(1 + \frac{\tau^2}{(\sigma+j)^2}\right)
\]

\[- (l+1) \log |\tau|.
\]
The first term in the right hand side of (12) is constant. The second term is bounded by \( C + (1 - \sigma) \log k \). Indeed,

\[
\sum_{j=0}^{k-1} \log \left( \frac{k-j}{k-j+\sigma-1} \right) = \sum_{j=0}^{k-1} \log \left( 1 + \frac{1 - \sigma}{k-j+\sigma-1} \right) = \sum_{j=0}^{k-1} \log \left( 1 + \frac{1 - \sigma}{l+\sigma+j} \right) \leq \log \left( 1 + \frac{1 - \sigma}{l+\sigma} \right) + \int_{1}^{k-1} \log \left( 1 + \frac{1 - \sigma}{x} \right) \, dx = C + x \log \left( 1 + \frac{1 - \sigma}{x} \right) \bigg|_{1}^{k-1} + (1 - \sigma) \int_{1}^{k-1} \frac{dx}{x(1 + \sigma)} \leq C + (1 - \sigma) \log k.
\]

To estimate the third term in the right hand side of (12) we note that the function \( \log \left( 1 + \frac{\tau^2}{(\sigma + j)^2} \right) \) is decreasing with respect to \( j \). Thus

\[
\frac{1}{2} \sum_{j=1}^{k-1} \log \left( 1 + \frac{\tau^2}{(\sigma + j)^2} \right) \geq \frac{1}{2} \int_{1}^{k} \log \left( 1 + \frac{\tau^2}{(\sigma + x)^2} \right) \, dx = \frac{1}{2} \int_{l+\sigma}^{k+\sigma} \log \left( 1 + \frac{\tau^2}{x^2} \right) \, dx = \frac{1}{2} (k + \sigma) \log \left( 1 + \frac{\tau^2}{(\sigma + k)^2} \right) - \frac{1}{2} (l + \sigma) \log \left( 1 + \frac{\tau^2}{(\sigma + l)^2} \right) + \tau \arctan \frac{k + \sigma}{\tau} - \tau \arctan \frac{l + \sigma}{\tau} + |\tau| \arctan \frac{k + \sigma}{|\tau|} \geq C + \frac{1}{2} (k + \sigma) \log \left( 1 + \frac{\tau^2}{(\sigma + k)^2} \right) - (l + \sigma) \log |\tau| + |\tau| \arctan \frac{k + \sigma}{|\tau|}.
\]
Finally

\[
\begin{align*}
    r(k) &\leq C + (1 + \sigma) \log k - \frac{1}{2} (k + \sigma) \log \left( 1 + \frac{\tau^2}{(\sigma + k)^2} \right) \\
    &\quad + (l + \sigma) \log |\tau| - |\tau| \arctan \frac{k + \sigma}{|\tau|} - (l + 1) \log |\tau| \\
    &= C + (1 - \sigma) \log k + (\sigma - 1) \log |\tau| \\
    &\quad - \frac{1}{2} (k + \sigma) \log \left( 1 + \frac{\tau^2}{(\sigma + k)^2} \right) - |\tau| \arctan \frac{k + \sigma}{|\tau|}.
\end{align*}
\]

(13)

Now we take \( k \sim \tau^2 \). Then the second term in the right hand side of (13) equals \( 2(1 - \sigma) \log |\tau| \) up to the additive constant. The fourth term is bounded. The fifth term equals

\[
|\tau| \arctan(|\tau| + o(1)) = \frac{\pi}{2} |\tau| + o(1).
\]

So

(14)

\[
r(k) \leq -\frac{\pi}{2} |\tau| + (1 - \sigma) \log |\tau| + C; k \sim \tau^2.
\]

Therefore after substitution \( [\tau^2] \) instead of \( k \) into (11) we obtain

\[
|b(s)| \leq C|\tau|^{2(\sigma - 1)} \exp \left( -\frac{\pi}{2} |\tau| + (1 - \sigma) \log |\tau| \right) = C|\tau|^{\sigma - 1} \exp \left( -\frac{\pi}{2} |\tau| \right).
\]

4. Estimation of \( |\zeta(\sigma + it)| \) for large \( |\tau| \)

Proposition 3. If operator \( A \) satisfies the assumptions of this paper then

\[
|\zeta(\sigma + it)| \leq C|\tau|^{\sigma - 1/2} \exp \left( \frac{\pi}{2} |\tau| \right), \quad C = C(\sigma), |\tau| \geq 1.
\]

Proof. We shall use the representation

(15)

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \theta(t) \, dt = \frac{F(s)}{\Gamma(s)}
\]

with

(16)

\[
\theta(t) = \text{Tr} e^{-tA} = \sum_{j=1}^{\infty} e^{-t\mu_j}.
\]

It is well known (e.g. see [2]) that

(17)

\[
\theta(t) \sim e^{-at}, \quad t \to \infty
\]

and (Minakshisundaram-Plejel expansion)

(18)

\[
\theta(t) \sim \sum_{j=-d}^{0} a_j t^{j/n} + \sum_{j=1}^{\infty} (a_j t^{j/n} + b_j t^{j/n} \log t), \quad t \to 0.
\]
Moreover one can differentiate both expansions (17) and (18). Split the integral from the right hand side of (15):

\[ J(s) = J_1(s) + J_2(s) = \int_0^1 t^{s-1} \theta(t) \, dt + \int_1^\infty t^{s-1} \theta(t) \, dt. \]

To estimate \( J_2 \) we use partial integration

\[
|J_2(s)| = \left| \frac{\theta(1)}{s} + \frac{1}{s} \int_1^\infty t^{s-1} \theta'(t) \, dt \right| \leq \frac{1}{|s|} \left[ \theta(1) + C \int_1^\infty t^s e^{-at} \, dt \right] \\
\leq \frac{C}{|\tau|}; \quad C = C(\sigma).
\]

To estimate \( J_1 \) we take \( l = 1 + ||\sigma|| \). Then

\[
\theta(t) = \sum_{j=-d}^l (a_j t^{j/n} + b_j t^{j/n} \log t) + r(t)
\]

\( (b_j = 0 \text{ if } j \leq 0) \) with

\[
|r(t)| \leq Ct^{(l+1)/n} |\log t| \quad \text{and} \quad |r'(t)| \leq Ct^{l/n} |\log t|; \quad 0 \leq t \leq 1.
\]

So

\[
J_1(s) = \sum_{j=-d}^l a_j \int_0^1 t^{j/n+s-1} \, dt + \sum_{j=1}^l b_j \int_0^1 t^{j/n+s-1} \log t \, dt + \int_0^1 t^{s-1} r(t) \, dt \\
= \sum_{j=-d}^l \frac{a_j}{s+j/n} - \sum_{j=1}^l \frac{b_j}{(s+j/n)^2} + \frac{1}{s} r(1) - \frac{1}{s} \int_0^1 t^{s-1} r'(t) \, dt
\]

and

\[
|J_1(s)| \leq C/|s| \leq C/|\tau|.
\]

Finally

\[
(19) \quad |J(s)| \leq \frac{C}{|\tau|}, \quad C = C(\sigma).
\]

To estimate \( 1/\Gamma(s) \) we use the Stirling asymptotics

\[
\log \Gamma(s) = (z - 1/2) \log z - z + (1/2) \log(2\pi) + o(1); \quad |z| \to \infty, |\arg z| < \pi
\]

(e.g. see [6]). We can write

\[
\log \Gamma(\sigma + i\tau) = [(\sigma - 1/2) + i\tau][(1/2) \log(\sigma^2 + \tau^2) + i \arg(\sigma + i\tau)] - \sigma - i\tau + (1/2) \log(2\pi) + o(1)
\]

and

\[
|\Gamma(\sigma + i\tau)| \sim \sqrt{2\pi} e^{-\sigma} \exp((1/2)(\sigma - 1/2) \log(\sigma^2 + \tau^2) - \tau \arg(\sigma + i\tau)); \quad |\tau| \to \infty.
\]

Note that

\[
\arg(\sigma + i\tau) = \frac{\pi}{2} \text{sgn } \tau - \frac{\sigma}{\tau} + o\left(\frac{1}{|\tau|}\right)
\]
and
\[ \log(\sigma^2 + \tau^2) = 2 \log|\tau| + o(1/|\tau|^2). \]

Therefore
\[ |\Gamma(\sigma + i\tau)| \sim \sqrt{2\pi|\tau|^{\sigma-1/2}} e^{-\pi|\tau|^{1/2}}; \quad |\tau| \to \infty. \]

Now the assertion of Proposition 3 follows from (15), (19) and (20).

5. Proof of the theorem

Propositions 2 and 3 show us that the function which is integrated in (8) is bounded by \( C|\tau|^{-3/2} \). So we can shift the path of integration to the left in the complex plane as far as we want. The poles of the function \( b(s)\zeta(s) \) give us terms in asymptotics of \( d^*(\omega) \). Denote
\[ \beta_j = \text{Res}_{b(s)}|_{s=-j}, \quad j = 0, 1, \ldots; \quad \hat{\beta}_0 = \text{Res}_{b(s)}|_{s=0}; \]
\[ \alpha_k = \text{Res}_{\zeta(s)}|_{s=-k/n}, \quad k = -d, -d+1, \ldots; \]
\[ \gamma_j = (\zeta(s) - \alpha_{nj}/(s+j))|_{s=-j}, \quad j = 1, 2, \ldots. \]

Note that \( \alpha_0 = 0 \) and if \( A \) is a differential operator or it is a power of a differential operator then
\[ \alpha_{nj} = 0 \quad \text{and} \quad \gamma_j = \zeta(-j). \]

By Proposition 2
\[ \beta_0 = 0 \quad \text{and} \quad \hat{\beta}_0 = 1. \]

The points \( s = -k/n, \quad k \neq jn(j = 0, 1, \ldots) \) are simple poles of the function \( F(s) = e^{s\omega}b(s)\zeta(s) \) and
\[ \text{Res}_{s=-k/n} F(s) = \alpha_k e^{-k\omega/n} b(-k/n). \]

The points \( s = jn \) are poles of the second order of \( F(s) \). It is easy to calculate residues in these points:
\[ \text{Res}_{s=0} F(s) = \zeta'(0) + \alpha \zeta(0), \]
\[ \text{Res}_{s=j} F(s) = \alpha_{nj}\beta_j \omega e^{-j\omega} + (\alpha_{nj}\hat{\beta}_j + \gamma_j\beta_j)e^{-j\omega}. \]

Thus we have obtained
\[ d^*(\omega) \sim \sum_{k=-d}^{\infty} p_k e^{-k\omega/n} + \sum_{j=0}^{\infty} q_j \omega e^{-j\omega}, \quad \omega \to \infty \]
with
\[ p_k = \alpha_k b(-k/n), k \neq jn(j = 0, 1, \ldots); \]
\[ p_0 = \zeta'(0) \]
\[ p_{jn} = \alpha_{nj}\hat{\beta}_j + \gamma_j\beta_j, \quad j = 1, 2, \ldots; \]
\[ q_0 = \zeta(0); \quad q_j = \alpha_{nj}\beta_j. \]

After substituting \( e^{-\omega} = \varepsilon \) into (22) one obtains (4). In particular (5) holds.
REFERENCES


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139