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TUTTE POLYNOMIALS AND BICYCLE DIMENSION OF TERNARY MATROIDS

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ABSTRACT. Let M be a ternary matroid, $t(M,x,y)$ be its Tutte polynomial and $d(M)$ be the dimension of the bicycle space of any representation of M over GF(3).

We show that, for $j = e^{2i\pi/3}$, the modulus of the complex number $t(M,j,j^2)$ is equal to $(\sqrt{3})^{d(M)}$. The proof relies on the study of the weight enumerator $W_{\mathscr{C}}(y)$ of the cycle space $\mathscr{C}$ of a representation of M over GF(3) evaluated at $y = j$. The main tool is the concept of principal quadripartition of $\mathscr{C}$ which allows a precise analysis of the evolution of the relevant invariants under deletion and contraction of elements.

RESUME. Soit M un matroïde ternaire, $t(M,x,y)$ son polynôme de Tutte et $d(M)$ la dimension de l'espace des bicycles d'une représentation quelconque de M sur GF(3).

Nous montrons que, pour $j = e^{2i\pi/3}$, le module du nombre complexe $t(M,j,j^2)$ est égal à $(\sqrt{3})^{d(M)}$. La preuve s'appuie sur l'étude de l'énumérateur de poids $W_{\mathscr{C}}(y)$ de l'espace des cycles $\mathscr{C}$ d'une représentation de M sur GF(3) pour la valeur $y = j$. L'outil essentiel est le concept de quadripartition principale de $\mathscr{C}$ qui permet une analyse précise de l'évolution des invariants concernés relativement à la suppression ou contraction d'éléments.

1. INTRODUCTION

The initial motivation for the present work was a discussion with Tom Brylawski (London, July 1987). He pointed out that, since all representations of a matroid M over F (where F is GF(2) or GF(3)) are essentially equivalent (see [2]), the dimension of the bicycle space (the intersection of the cycle and cocycle spaces) of any such representation depends only on M. Let us denote (when M is representable over F) this dimension by $d(M,F)$. When F is GF(2), $d(M,F)$ can be obtained from the Tutte polynomial $t(M,x,y)$ of M. More precisely, if M is a binary matroid on the set E, $t(M,-1,-1) = (-1)^{|E|}(-2)^{d(M,\mathrm{GF}(2))}$ (this is proved in [8, Theorem 9.1]

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for graphic matroids, and the extension of the proof to arbitrary binary matroids is immediate). Brylawski asked whether a similar formula existed for $d(M, \mathrm{GF}(3))$. The present paper gives a positive answer to this question.

The essential clue in this research was the realization that a knot-theoretic result of Lickorish and Millett [6, Theorem 3], restricted to alternating links, gives the solution of Brylawski's question for cycle matroids of plane graphs. This result relates the Jones polynomial $V_L(t)$ of a link L evaluated at $t = e^{i\pi/3}$ to the dimension of the GF(3)-homology $H_1(D_L, \mathrm{GF}(3))$ of the double cyclic cover of S^3 branched along L. For alternating links, the Jones polynomial is closely related to the Tutte polynomial of an associated plane graph G (see for instance [5], [9]), and (using an appropriate presentation matrix) one can also interpret $H_1(D_L, \mathrm{GF}(3))$ as the bicycle space of G over GF(3). In fact it can be checked that the restriction of our main result to cycle matroids of plane graphs is equivalent to the restriction of the Lickorish-Millett result to alternating links. A detailed account of this connection between the two results, whose proofs are entirely different, is beyong the scope of the present paper.

The paper is organized as follows. In Section 2 we work in the context of spaces over GF(3) and no matroid-theoretic knowledge is required. We introduce the principal quadripartition of a ternary space and use it to study the modification of the bicycle dimension under deletion and contraction of elements. Then we relate the bicycle dimension of a ternary space to a special value of its weight enumerator. Section 3 reformulates the previous results in matroid-theoretic terms, using Greene's formula [4] which relates the weight enumerator of a space to the Tutte polynomial of the associated matroid. Finally some concluding remarks are presented in Section 4.

2. Bicycle dimension and weight enumeration in ternary spaces

2.1 **Definitions, notations.** Let E be a finite nonempty set. The set $\mathrm{GF}(3)^E$ of mappings from E to GF(3) is endowed with its canonical structure of vector space over GF(3). For every mapping C in $\mathrm{GF}(3)^E$ the *support* of C, denoted by $S(C)$, is the set of elements e of E such that $C(e) \neq 0$. We shall denote $|S(C)|$ by $s(C)$. For every element e of E we denote by $\underline{e}$ the mapping from E to GF(3) which takes the value 1 on e and the value 0 on $E - \{e\}$. For C and K in $\mathrm{GF}(3)^E$ we write $\langle C, K\rangle = \sum_{e\in E} C(e)K(e)$, and C, K are said to be *orthogonal* if $\langle C, K\rangle = 0$.

We call *ternary space on* E, or more simply in this section *space on* E, any subspace of $\mathrm{GF}(3)^E$. For any space $\mathscr{C}$ on E, we denote by $\mathscr{C}^\perp$ the space on E formed by the elements of $\mathrm{GF}(3)^E$ orthogonal to all elements of $\mathscr{C}$, by $\mathscr{B}(\mathscr{C})$ the space $\mathscr{C} \cap \mathscr{C}^\perp$ on E and by $d(\mathscr{C})$ the dimension of $\mathscr{B}(\mathscr{C})$, which we call the *bicycle dimension* of $\mathscr{C}$. A *loop* of $\mathscr{C}$ is an element e of E such that $\underline{e} \in \mathscr{C}$, and a *coloop* of $\mathscr{C}$ is a loop of $\mathscr{C}^\perp$. For $|E| \geq 2$ and e in E, we denote by $\mathscr{C} - e$ the space $\{C \cdot e / C \in \mathscr{C}, C(e) = 0\}$ on $E - \{e\}$ and by $\mathscr{C} \cdot e$ the space $\{C \cdot e / C \in \mathscr{C}\}$ on $E - \{e\}$, where $C \cdot e$ denotes the

restriction of C to $E-\{e\}$. Then it is easy to check that $(\mathscr{C}-e)^{\perp}=\mathscr{C}^{\perp}\cdot e$ and $(\mathscr{C}\cdot e)^{\perp}=\mathscr{C}^{\perp}-e$. In the sequel whenever the notations $\mathscr{C}-e$ and $\mathscr{C}\cdot e$ are used it is implicitly assumed that $|E|\geq 2$.

2.2 **The principal quadripartition and bicycle dimension.** Let E be a finite nonempty set and $\mathscr{C}$ be a space on E. The following result is analogous to the tripartition theorem given in [8, Theorem 2.1] for binary cycle spaces of graphs (and it can obviously be extended to spaces over any finite field).

Proposition 1. *For every element e of E exactly one of the following properties holds*:

(1) *there exists B in $\mathscr{B}(\mathscr{C})$ with $B(e)\neq 0$*;
(2a) *there exists C in $\mathscr{C}$ and K in $\mathscr{C}^{\perp}$ with $\underline{e}=C+K$ and $C(e)=1$, $K(e)=0$*;
(2b) *there exists C in $\mathscr{C}$ and K in $\mathscr{C}^{\perp}$ with $\underline{e}=C+K$ and $C(e)=0$, $K(e)=1$*;
(2c) *there exists C in $\mathscr{C}$ and K in $\mathscr{C}^{\perp}$ with $\underline{e}=C+K$ and $C(e)=-1$, $K(e)=-1$.*

Proof. Consider the property: (2) $\underline{e}\in\mathscr{C}+\mathscr{C}^{\perp}$. Since $(\mathscr{C}+\mathscr{C}^{\perp})^{\perp}=\mathscr{C}^{\perp}\cap(\mathscr{C}^{\perp})^{\perp}=\mathscr{C}^{\perp}\cap\mathscr{C}$, (2) holds if and only if $\underline{e}$ is orthogonal to every B in $\mathscr{B}(\mathscr{C})$. Hence (2) is the negation of (1). On the other hand, (2) holds if and only if (2a), (2b) or (2c) holds. Thus it remains to show that (2a), (2b) and (2c) are mutually exclusive. Assume that (2) holds and that $\underline{e}=C+K=C'+K'$ with C, C' in $\mathscr{C}$ and K,K' in $\mathscr{C}^{\perp}$. Then $C-C'=K'-K$ is an element B of $\mathscr{B}(\mathscr{C})$. Since (1) does not hold, $B(e)=0$ and hence $C(e)=C'(e)$ and $K(e)=K'(e)$. This completes the proof. □

We shall say that the element e of E is *of type* (1), (2a), (2b), *or* (2c) (*with respect to* $\mathscr{C}$) if e satisfies the corresponding property in Proposition 1.

Remark. A loop is of type (2a) and a coloop is of type (2b).

Proposition 2. *If e is a loop or coloop, $d(\mathscr{C}-e)=d(\mathscr{C}\cdot e)=d(\mathscr{C})$. Otherwise*:
(1) *If e is of type* (1), *then $d(\mathscr{C}-e)=d(\mathscr{C})-1$ and $d(\mathscr{C}\cdot e)=d(\mathscr{C})-1$.*
(2a) *If e is of type* (2a), *then $d(\mathscr{C}-e)=d(\mathscr{C})$ and $d(\mathscr{C}\cdot e)=d(\mathscr{C})+1$.*
(2b) *If e is of type* (2b), *then $d(\mathscr{C}-e)=d(\mathscr{C})+1$ and $d(\mathscr{C}\cdot e)=d(\mathscr{C})$.*
(2c) *If e is of type* (2c), *then $d(\mathscr{C}-e)=d(\mathscr{C})$ and $d(\mathscr{C}\cdot e)=d(\mathscr{C})$.*

Proof. First note that $\mathscr{B}(\mathscr{C}-e)=(\mathscr{C}-e)\cap(\mathscr{C}-e)^{\perp}=(\mathscr{C}-e)\cap\mathscr{C}^{\perp}\cdot e=\{K\cdot e/K\in\mathscr{C}^{\perp},(K\cdot e)^{\wedge}\in\mathscr{C}\}$, where $(K\cdot e)^{\wedge}$ is the mapping from E to GF(3) which coincides with $K\cdot e$ on $E-\{e\}$ and takes the value 0 on e. Clearly $\mathscr{B}(\mathscr{C})-e=\{B\cdot e/B\in\mathscr{C}^{\perp},B\in\mathscr{C},B(e)=0\}$ is a subspace of $\mathscr{B}(\mathscr{C}-e)$. An element of $\mathscr{B}(\mathscr{C}-e)-(\mathscr{B}(\mathscr{C})-e)$ is of the form $K\cdot e$ for some $K\in\mathscr{C}^{\perp}$ with $(K\cdot e)^{\wedge}\in\mathscr{C}$ and $K(e)\neq 0$. Since $\underline{e}=K(e)K-K(e)(K\cdot e)^{\wedge}$, e must then be of type (2b) with respect to $\mathscr{C}$.

Thus if e is not of type (2b), $\mathscr{B}(\mathscr{C}-e)=\mathscr{B}(\mathscr{C})-e$. Hence $d(\mathscr{C}-e)=\dim\mathscr{B}(\mathscr{C})\cap\mathscr{H}$, where $\mathscr{H}$ is the hyperplane orthogonal to $\underline{e}$. It follows that $d(\mathscr{C}-e)=d(\mathscr{C})-1$ if e is of type (1), and $d(\mathscr{C}-e)=d(\mathscr{C})$ if e is of type (2a) or (2c).

If e is of type (2b) there exists C_0 in $\mathscr{C}$ and K_0 in $\mathscr{C}^\perp$ with $\underline{e}=C_0+K_0$ and $C_0(e)=0$, $K_0(e)=1$. Then $(K_0\cdot e)^\wedge=-C_0$ belongs to $\mathscr{C}$ and hence $K_0\cdot e$ belongs to $\mathscr{B}(\mathscr{C}-e)$. Moreover if $K\cdot e$ belongs to $\mathscr{B}(\mathscr{C}-e)-(\mathscr{B}(\mathscr{C})-e)$ (so that $K\in\mathscr{C}^\perp$, $(K\cdot e)^\wedge\in\mathscr{C}$ and $K(e)\neq 0$), $K(e)K-K(e)(K\cdot e)^\wedge=\underline{e}=C_0+K_0$. Hence $B=K(e)K-K_0=C_0+K(e)(K\cdot e)^\wedge$ belongs to $\mathscr{B}(\mathscr{C})$ and takes the value zero on e. Then $K(e)K\cdot e-K_0\cdot e=B\cdot e$ belongs to $\mathscr{B}(\mathscr{C})-e$. It follows that $\mathscr{B}(\mathscr{C}-e)=(\mathscr{B}(\mathscr{C})-e)+\mathscr{S}$, where $\mathscr{S}$ is the space generated by $K_0\cdot e$. Note that $K_0\cdot e$ belongs to $(\mathscr{B}(\mathscr{C})-e)$ if and only if $K_0\cdot e=B\cdot e$ for some $B\in\mathscr{B}(\mathscr{C})$ with $B(e)=0$. In this case $\underline{e}=K_0-B$ belongs to $\mathscr{C}^\perp$ and hence e is a coloop of $\mathscr{C}$. Conversely if e is a coloop of $\mathscr{C}$, $C_0=\underline{e}-K_0$ belongs to $\mathscr{B}(\mathscr{C})$, $C_0(e)=0$ and $K_0\cdot e=-C_0\cdot e$, so that $K_0\cdot e$ belongs to $(\mathscr{B}(\mathscr{C})-e)$. It follows that $d(\mathscr{C}-e)=\dim(\mathscr{B}(\mathscr{C})-e)$ if e is a coloop, and $d(\mathscr{C}-e)=\dim(\mathscr{B}(\mathscr{C})-e)+1$ otherwise. Since e is not of type (1), $\dim(\mathscr{B}(\mathscr{C})-e)=\dim\mathscr{B}(\mathscr{C})\cap\mathscr{H}=\dim\mathscr{B}(\mathscr{C})$. This completes the proof of the evaluation of $d(\mathscr{C}-e)$.

For the evaluation of $d(\mathscr{C}\cdot e)$, we observe that $d(\mathscr{C})=d(\mathscr{C}^\perp)$, $d(\mathscr{C}\cdot e)=d((\mathscr{C}\cdot e)^\perp)=d(\mathscr{C}^\perp-e)$ and we apply the above results to $\mathscr{C}^\perp$. We also use the obvious facts that e is of type (1) (respectively: (2a), (2b) (2c)) with respect to $\mathscr{C}$ if and only if it is of type (1) (respectively: (2b), (2a), (2c)) with respect to $\mathscr{C}^\perp$, and that e is a loop (respectively: coloop) of $\mathscr{C}$ if and only if it is a coloop (respectively: loop) of $\mathscr{C}^\perp$. □

2.3 **The main result.** Let $j=e^{2i\pi/3}$, so that $j^3=1$ and $1+j+j^2=0$. To every space $\mathscr{C}$ on the finite nonempty set E we associate the complex number $f(\mathscr{C})=\sum_{C\in\mathscr{C}}j^{s(C)}$, which is the evaluation at $y=j$ of the weight enumerator $W_{\mathscr{C}}(y)$ of $\mathscr{C}$ considered as a linear code (see [7, Chapter 5]).

Proposition 3. *Let $\mathscr{C}$ be a space on E and e be an element of E of type (2b) with respect to $\mathscr{C}$. Then $f(\mathscr{C})=f(\mathscr{C}\cdot e)$.*

Proof. Clearly

$$f(\mathscr{C})=\sum_{C\in\mathscr{C},C(e)=0}j^{s(C)}+2\sum_{C\in\mathscr{C},C(e)=1}j^{s(C)}.$$

Since e is not a loop, every element of $\mathscr{C}\cdot e$ is of the form $C\cdot e$ for a unique element C of $\mathscr{C}$ (because if $C\cdot e=C'\cdot e$, $S(C-C')\subseteq\{e\}$ implies $C=C'$). It follows that:

$$f(\mathscr{C}\cdot e)=\sum_{C\in\mathscr{C}}j^{s(C\cdot e)}=\sum_{C\in\mathscr{C},C(e)=0}j^{s(C)}+2\sum_{C\in\mathscr{C},C(e)=1}j^{s(C)-1}.$$

We now prove that $g(\mathscr{C})=\sum_{C\in\mathscr{C},C(e)=1}j^{s(C)}=0$. Let Z be an element of $\mathscr{C}$ with $Z(e)=0$ such that $\underline{e}=Z+K$ for some K in $\mathscr{C}^\perp$. Then, since

$\{C \in \mathscr{C}, C(e) = 1\}$ is invariant under a translation of vector Z:

$$3g(\mathscr{C}) = \sum_{C \in \mathscr{C}, C(e)=1} j^{s(C)} + j^{s(C+Z)} + j^{s(C-Z)}.$$

For every $C \in \mathscr{C}$ we write $C_0 = \{x \in E/C(x) \neq 0,\ Z(x) = 0\}$, $C_+ = \{x \in E/C(x) = Z(x) \neq 0\}$, $C_- = \{x \in E/C(x) = -Z(x) \neq 0\}$, $Z_0 = \{x \in E/C(x) = 0,\ Z(x) \neq 0\}$. Then $s(C) = |C_0| + |C_+| + |C_-|$, $S(C+Z) = |C_0| + |C_+| + |Z_0|$ and $s(C-Z) = |C_0| + |C_-| + |Z_0|$.

Let C be an element of $\mathscr{C}$ with $C(e) = 1$. Since $0 = \langle C, K\rangle = \langle C, \underline{e} - Z\rangle = 1 - \langle C, Z\rangle$ we have $|C_+| - |C_-| \equiv 1 \pmod 3$. Similarly, since $0 = \langle Z, K\rangle = \langle Z, \underline{e} - Z\rangle = -\langle Z, Z\rangle$ we have $s(Z) = |Z_0| + |C_+| + |C_-| \equiv 0 \pmod 3$.

Now $j^{s(C)} + j^{s(C+Z)} + j^{s(C-Z)} = j^{|C_0|}(j^{|C_+|+|C_-|} + j^{|C_+|+|Z_0|} + j^{|C_-|+|Z_0|})$. Since $|C_+| + |C_-| \equiv 2|C_-| + 1 \pmod 3$, $|C_+| + |Z_0| \equiv 2|C_-| \pmod 3$, $|C_-| + |Z_0| \equiv 2|C_-| - 1 \pmod 3$ and $1 + j + j^2 = 0$, $j^{s(C)} + j^{s(C+Z)} + j^{s(C-Z)} = 0$. It follows that $g(\mathscr{C}) = 0$ as required. □

Proposition 4. *Let $\mathscr{C}$ be a space on E and e be an element of E which is not a loop of $\mathscr{C}$. Then*

(1) *If e is of type* (1) *with respect to $\mathscr{C}$, $f(\mathscr{C}) = (j - j^2)f(\mathscr{C} \cdot e)$.*
(2a) *If e is of type* (2a) *with respect to $\mathscr{C}$, $f(\mathscr{C}) = ((j - j^2)/3)f(\mathscr{C} \cdot e)$.*
(2c) *If e is of type* (2c) *with respect to $\mathscr{C}$, $f(\mathscr{C}) = -f(\mathscr{C} \cdot e)$.*

Proof. Let Z be an element of $\mathscr{C}$ with $Z(e) \neq 0$ (which exists since e is not of type (2b) and hence not a coloop), $\mathscr{Z}$ be the space generated by Z, and $\mathscr{H}$ be the hyperplane orthogonal to $\underline{e}$. Then clearly $\mathscr{C} = (\mathscr{C} \cap \mathscr{H}) \oplus \mathscr{Z}$, where $\oplus$ denotes the direct sum of vector spaces. It follows that:

$$f(\mathscr{C}) = \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C)} + j^{s(C+Z)} + j^{s(C-Z)}.$$

Now, since every element of $\mathscr{C} \cdot e$ is of the form $C \cdot e$ for a unique element C of $\mathscr{C}$, we may write:

$$\begin{aligned} f(\mathscr{C} \cdot e) &= \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C \cdot e)} + j^{s((C+Z) \cdot e)} + j^{s((C-Z) \cdot e)} \\ &= \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C)} + j^{s(C+Z)-1} + j^{s(C-Z)-1}. \end{aligned}$$

For every C in $\mathscr{C}$ we define C_0, C_+, C_-, Z_0 as in the proof of Proposition 3, so that $s(C) = |C_0| + |C_+| + |C_-|$, $s(C+Z) = |C_0| + |C_+| + |Z_0|$ and $s(C-Z) = |C_0| + |C_-| + |Z_0|$. It follows that

(i) $f(\mathscr{C}) = \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C)}(1 + j^{|Z_0| - |C_-|} + j^{|Z_0| - |C_+|})$;
(ii) $f(\mathscr{C} \cdot e) = \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C)}(1 + j^{|Z_0| - |C_-| - 1} + j^{|Z_0| - |C_+| - 1})$.

(1). If e is of type (1) with respect to $\mathscr{C}$, we take for Z an element of $\mathscr{C} \cap \mathscr{C}^{\perp}$ with $Z(e) \neq 0$. Then for any C in $\mathscr{C} \cap \mathscr{H}$, Z is orthogonal to C and hence

$|C_+| \equiv |C_-| \pmod 3$. Moreover, since Z is orthogonal to itself, $|Z_0| + |C_+| + |C_-| \equiv 0 \pmod 3$ and hence $|Z_0| \equiv |C_+| \equiv |C_-| \pmod 3$. It follows that $1 + j^{|Z_0|-|C_-|} + j^{|Z_0|-|C_+|} = 3$ and $1 + j^{|Z_0|-|C_-|-1} + j^{|Z_0|-|C_+|-1} = 1 + 2j^2$. Since $3 = (j - j^2)(1 + 2j^2)$, it follows from (i) and (ii) that $f(\mathscr{C}) = (j - j^2) f(\mathscr{C} \cdot e)$.

$(2a)$. If e is of type $(2a)$ with respect to $\mathscr{C}$, we take for Z an element of $\mathscr{C}$ with $Z(e) = 1$ and $\underline{e} = Z + K$ for some K in $\mathscr{C}^\perp$. Then for any C in $\mathscr{C} \cap \mathscr{H}$, $0 = \langle C, K\rangle = \langle C, \underline{e} - Z\rangle = -\langle C, Z\rangle$ and hence $|C_+| \equiv |C_-| \pmod 3$. Similarly, since $0 = \langle Z, K\rangle = \langle Z, \underline{e} - Z\rangle = 1 - \langle Z, Z\rangle$, $|Z_0| + |C_+| + |C_-| \equiv 1 \pmod 3$ and hence $|Z_0| - |C_+| \equiv |Z_0| - |C_-| \equiv 1 \pmod 3$. It follows that $1 + j^{|Z_0|-|C_-|} + j^{|Z_0|-|C_+|} = 1 + 2j = j - j^2$ and $1 + j^{|Z_0|-|C_-|-1} + j^{|Z_0|-|C_+|-1} = 3$. Hence, by (i) and (ii), $f(\mathscr{C}) = ((j - j^2)/3) f(\mathscr{C} \cdot e)$.

$(2c)$. If e is of type $(2c)$ with respect to $\mathscr{C}$, we take for Z an element of $\mathscr{C}$ with $Z(e) = -1$ and $\underline{e} = Z + K$ for some K in $\mathscr{C}^\perp$. As before, for any C in $\mathscr{C} \cap \mathscr{H}$, $|C_+| \equiv |C_-| \pmod 3$. Moreover $0 = \langle Z, \underline{e} - Z\rangle = -1 - \langle Z, Z\rangle$ implies $|Z_0| + |C_+| + |C_-| \equiv 2 \pmod 3$ and hence $|Z_0| - |C_+| \equiv |Z_0| - |C_-| \equiv 2 \pmod 3$. It follows that $1 + j^{|Z_0|-|C_-|} + j^{|Z_0|-|C_+|} = 1 + 2j^2$ and $1 + j^{|Z_0|-|C_-|-1} + j^{|Z_0|-|C_+|-1} = 1 + 2j = -1 - 2j^2$. Hence, by (i) and (ii), $f(\mathscr{C}) = -f(\mathscr{C} \cdot e)$. □

For every space $\mathscr{C}$ on E, let $\varepsilon(\mathscr{C}) = (j - j^2)^{-\dim \mathscr{C} - d(\mathscr{C})} f(\mathscr{C})$.

Proposition 5. *For every space $\mathscr{C}$ on E, $\varepsilon(\mathscr{C}) = \varepsilon(\mathscr{C} \cdot e)$ if e is a loop or is of type (1) or $(2b)$ with respect to $\mathscr{C}$, and $\varepsilon(\mathscr{C}) = -\varepsilon(\mathscr{C} \cdot e)$ otherwise.*

Proof. If e is a loop, $\mathscr{C} = (\mathscr{C} \cap \mathscr{H}) \oplus \mathscr{Z}$, where $\mathscr{Z}$ is the space generated by $\underline{e}$ and $\mathscr{H}$ is the hyperplane orthogonal to $\underline{e}$. Hence $f(\mathscr{C}) = \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C)} + j^{s(C+\underline{e})} + j^{s(C-\underline{e})}$ and $f(\mathscr{C} \cdot e) = \sum_{C \in \mathscr{C} \cap \mathscr{H}} j^{s(C)}$. It follows that $f(\mathscr{C}) = (1 + 2j) f(\mathscr{C} \cdot e) = (j - j^2) f(\mathscr{C} \cdot e)$. Clearly $\dim \mathscr{C} = \dim(\mathscr{C} \cdot e) + 1$ and, by Proposition 2, $d(\mathscr{C}) = d(\mathscr{C} \cdot e)$. Hence $\varepsilon(\mathscr{C}) = \varepsilon(\mathscr{C} \cdot e)$.

If e is not a loop, $\dim \mathscr{C} = \dim(\mathscr{C} \cdot e)$ and the result follows easily from Propositions 2, 3, 4. □

Proposition 6. *For every space $\mathscr{C}$ on E, $\sum_{C \in \mathscr{C}} j^{s(C)} = f(\mathscr{C}) = \varepsilon(\mathscr{C})(j - j^2)^{\dim \mathscr{C} + d(\mathscr{C})}$ with $\varepsilon(\mathscr{C}) \in \{1, -1\}$. Equivalently, $(f(\mathscr{C}))^2 = (-3)^{\dim \mathscr{C} + d(\mathscr{C})}$.*

Proof. If $|E| = 1$ and e is a loop of $\mathscr{C}$, $\dim \mathscr{C} = 1$, $d(\mathscr{C}) = 0$ and $f(\mathscr{C}) = 1 + 2j = j - j^2$, so that $\varepsilon(\mathscr{C}) = 1$. If $|E| = 1$ and e is a coloop of $\mathscr{C}$, $\dim \mathscr{C} = 0$, $d(\mathscr{C}) = 0$ and $f(\mathscr{C}) = 1$, so that $\varepsilon(\mathscr{C}) = 1$ also in this case. Then the result follows by induction using Proposition 5. □

3. Bicycle dimension and Tutte polynomials

Now we need some basic notions of matroid theory which can be found in [12]. Let E be a finite nonempty set, $\mathscr{C}$ be a subspace of $\mathrm{GF}(q)^E$ and $S(\mathscr{C})$

be the set of supports of elements of $\mathscr{C}$ (the definitions and notations are the same as those at the beginning of Section 2). The elements of $S(\mathscr{C})-\{\varnothing\}$ which are minimal by inclusion form the circuits of a matroid on E which we denote by $M(\mathscr{C})$. Then $\mathscr{C}$ is the cycle space of an appropriate representation R of $M(\mathscr{C})$ over $\mathrm{GF}(q)$, $\mathscr{C}^{\perp}$ is the cocycle space of R and $M(\mathscr{C})$ and $M(\mathscr{C}^{\perp})$ are dual matroids. The vector space $\mathscr{C}\cap\mathscr{C}^{\perp}$ is usually called the *bicycle space* of the representation R. A matroid is said to be *ternary* if it is of the form $M(\mathscr{C})$ for some ternary space $\mathscr{C}$.

Let M be a matroid on E with rank-function r. The *Tutte polynomial of* M (first introduced for graphic matroids in [10], [11] and then for matroids in [1] and [3]) is the polynomial $t(M,x,y)$ in two variables x,y defined by

$$t(M,x,y)=\sum_{F\subseteq E}(x-1)^{r(E)-r(F)}(y-1)^{|F|-r(F)}.$$

This polynomial satisfies the following properties which yield a recursive definition:

(i) If $|E|=1$: if the unique element of E is a coloop of M, $t(M,x,y)=x$; if it is a loop of M, $t(M,x,y)=y$.

(ii) If the element e of E is neither a loop nor a coloop of M, $t(M,x,y)=t(M-e,x,y)+t(M\cdot e,x,y)$, where $M-e$ (respectively: $M\cdot e$) denotes the matroid obtained from M by the deletion (respectively: contraction) of e.

(iii) If M is the direct sum of the matroids M_1 and M_2, $t(M,x,y)=t(M_1,x,y)t(M_2,x,y)$.

Another property of interest is that if M and M^* are dual matroids, $t(M^*,x,y)=t(M,y,x)$.

The following formula is due to Greene [4]. We shall provide a short proof for the sake of completeness.

Proposition 7. *Let $\mathscr{C}$ be a subspace of* $\mathrm{GF}(q)^{|E|}$ *and r be the rank-function of* $M(\mathscr{C})$.

(i) $\sum_{K\in\mathscr{C}^{\perp}} z^{s(K)}=z^{|E|-r(E)}(1-z)^{r(E)}t(M(\mathscr{C}),(1+(q-1)z)/(1-z),1/z)$.

(i*) $\sum_{C\in\mathscr{C}} z^{s(C)}=z^{r(E)}(1-z)^{|E|-r(E)}t(M(\mathscr{C}),1/z,(1+(q-1)z)/(1-z))$.

Proof. Consider the polynomial $P(v)=\sum_{K\in\mathscr{C}^{\perp}}(1+v)^{|E|-s(K)}$ in the variable v. Clearly

$$P(v)=\sum_{K\in\mathscr{C}^{\perp}}\sum_{F\subseteq E-S(K)}v^{|F|}=\sum_{F\subseteq E}\sum_{K\in\mathscr{C}^{\perp},S(K)\subseteq E-F}v^{|F|}.$$

Let I be a spanning independent subset of F and B be a basis of $M(\mathscr{C})$ which contains I. For $K\in\mathscr{C}^{\perp}$, $S(K)\subseteq E-F$ iff the restriction of K to I is zero. Considering the fundamental basis of the space $\mathscr{C}^{\perp}$ associated to B it is clear that the number of elements $K\in\mathscr{C}^{\perp}$ such that $S(K)\subseteq E-F$ is

$q^{|B|-|I|} = q^{r(E)-r(F)}$. Hence $P(v) = \sum_{F \subseteq E} q^{r(E)-r(F)} v^{|F|}$

$$= v^{r(E)} \sum_{F \subseteq E} (q/v)^{r(E)-r(F)} v^{|F|-r(F)} = v^{r(E)} t(M(\mathscr{C}), 1 + (q/v), 1 + v).$$

Taking $1 + v = 1/z$ yields formula (i). Formula (i*) follows by duality. □

Now, coming back to ternary matroids, let $\mathscr{C}$ be a subspace of $\mathrm{GF}(3)^{|E|}$ and r be the rank-function of $M(\mathscr{C})$. Taking $q = 3$ and $z = j$ in formula (i*) of Proposition 7 yields $f(\mathscr{C}) = \sum_{C \in \mathscr{C}} j^{s(C)} = j^{r(E)}(1-j)^{|E|-r(E)} t(M(\mathscr{C}), j^2, j)$. Comparing with Proposition 6 and noting that $|E| - r(E) = \dim \mathscr{C}$, we obtain:

$$j^{|E|-\dim \mathscr{C}}(1-j)^{\dim \mathscr{C}} t(M(\mathscr{C}), j^2, j) = \varepsilon(\mathscr{C})(j - j^2)^{\dim \mathscr{C} + d(\mathscr{C})}.$$

This formula and its complex conjugate yield:

Proposition 8. *For any ternary space $\mathscr{C}$ on E:*

(i) $t(M(\mathscr{C}), j^2, j) = \varepsilon(\mathscr{C})(j^2)^{\dim \mathscr{C} + |E|}(j - j^2)^{d(\mathscr{C})}$;

(ii) $t(M(\mathscr{C}), j, j^2) = \varepsilon(\mathscr{C}) j^{\dim \mathscr{C} + |E|}(j^2 - j)^{d(\mathscr{C})}$ *with* $\varepsilon(\mathscr{C}) \in \{1, -1\}$. *The modulus of* $t(M(\mathscr{C}), j, j^2)$ *is* $(\sqrt{3})^{d(\mathscr{C})}$.

4. Concluding remarks

The unicity of ternary representations [2] allows us to associate to every ternary matroid M the dimension $d(M)$ of the bicycle space for any representation of M over $\mathrm{GF}(3)$. By Proposition 8, $d(M)$ depends only on the Tutte polynomial of M.

In fact we may define a signature mapping ε from the class of all ternary matroids to $\{1, -1\}$ by setting $\varepsilon(M(\mathscr{C})) = \varepsilon(\mathscr{C})$ for every ternary space $\mathscr{C}$ (the consistency of this definition follows from Proposition 8). We can also define the quadripartition of M as follows: the type of the element e is $(2a)$ if it is a loop, $(2b)$ if it is a coloop, and otherwise depends on the values of the invariant d on M, $M - e$ and $M \cdot e$ as specified in Proposition 2.

Proposition 9. *For every ternary matroid M on E with dual M^*:*

(i) $\varepsilon(M) = (-1)^{d(M)} \varepsilon(M^*)$;

(ii) *if* $|E| \geq 2$, $\varepsilon(M) = \varepsilon(M \cdot e)$ *if e is a loop or is of type* (1) *or* $(2b)$ *with respect to M, and* $\varepsilon(M) = -\varepsilon(M \cdot e)$ *otherwise;*

(iii) *if* $|E| \geq 2$, $\varepsilon(M) = \varepsilon(M - e)$ *if and only if e is of type* $(2a)$ *or* $(2b)$ *with respect to M.*

Proof. Noting that $t(M, j, j^2)$ and $t(M^*, j, j^2)$ are complex conjugates, we easily deduce (i) from Proposition 8. Statement (ii) is just a reformulation of Proposition 5. Finally (iii) is easily obtained from (ii) by duality, using (i) and Proposition 2. □

It is not difficult to see that Proposition 8 follows by induction (using the recursive definition of the Tutte polynomial given in Section 3) from Proposition

9. Our proof of the existence of a signature ε satisfying Proposition 9 uses the weight enumerator f. It would be interesting to have a more direct proof, either by showing the consistency of (i), (ii), (iii) considered as axioms, or by finding a simpler interpretation of ε.

Finally we note that a simple necessary condition for a matroid M to be ternary is that the modulus of $t(M, j, j^2)$ be of the form $(\sqrt{3})^d$ for some integer d with $0 \leq d \leq \min\{r(M), r(M^*)\}$, where r denotes rank.

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