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# THE LINEAR AND QUADRATIC SCHUR SUBGROUPS OVER THE $S$-INTEGERS OF A NUMBER FIELD

CARL R. RIEHM

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ABSTRACT. Let $K$ be an algebraic number field and let $\mathfrak{O}$ be a ring of $S$-integers in $K$ (where $S$ is a set of primes of $K$ containing all the archimedean primes); that is to say, $\mathfrak{O}$ is a Dedekind domain whose field of quotients is $K$. In analogy with a theorem of T. Yamada in the case of a field of characteristic $0$, it is shown that if $S(\mathfrak{O})$ is the Schur subgroup of the Brauer group $B(\mathfrak{O})$ and if $\mathfrak{o} = \mathfrak{O} \cap k$, where $k$ is any field containing the maximal abelian extension of $\mathbb{Q}$ in $K$, then $S(\mathfrak{O}) = \mathfrak{O} \otimes S(\mathfrak{o})$, i.e. the Brauer classes in $S(\mathfrak{O})$ are those obtained from $S(\mathfrak{o})$ by extension of the scalars to $\mathfrak{O}$. A similar theorem is proved as well in the case of the Schur subgroup $S(\mathfrak{O}, \omega)$ of the quadratic Brauer group $B(\mathfrak{O}, \omega)$, where $\omega$ is an involution of $\mathfrak{O}$.

Suppose that $K$ is an algebraic number field (of finite degree over $\mathbb{Q}$) and let $\mathfrak{O}$ be the ring of $S$-integers of $K$ where $S$ is a set of primes of $K$ containing the infinite ones. Thus $\mathfrak{O}$ is a Dedekind domain whose field of quotients is $K$. Let $B(\mathfrak{O})$ denote the (linear) Brauer group of $\mathfrak{O}$. The *Schur subgroup* $S(\mathfrak{O})$ of $B(\mathfrak{O})$ consists of the Brauer classes which contain an Azumaya algebra $\Lambda$ over $\mathfrak{O}$ with the following property: there exists a finite group $G$ and a central simple direct summand $A$ of the group algebra $KG$ such that the image of the projection of $\mathfrak{O}G$ in $A$ is $\Lambda$.

Let $\mathbb{Q}(\mu)$ denote the maximal cyclotomic extension of $\mathbb{Q}$ ($\mu$ is the group of all roots of unity), contained in an algebraic closure of $K$. Denote by $K_c$ the intersection of $\mathbb{Q}(\mu)$ and $K$, and let $K \otimes S(K_c)$ be the subgroup of $B(K)$ obtained from $S(K_c)$ by extension of scalars. It is clear that $K \otimes S(K_c) \subseteq S(K)$—in fact, they are equal by a theorem of Yamada [Prop. 4.6, Y]; see also [Theorem 3.4, M]). We shall prove the following integral analogue:

**Theorem 1.** *Let $k$ be a subfield of $K$ containing $K_c$, and let $\mathfrak{o} = \mathfrak{O} \cap k$. Then*

$$S(\mathfrak{O}) = \mathfrak{O} \otimes S(\mathfrak{o}).$$

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This effectively reduces the computation of $S(\mathfrak{O})$ to that of $S(\mathfrak{o})$, since $S(\mathfrak{o})$ and $S(\mathfrak{O})$ are subgroups of $B(k)$ and $B(K)$ resp., and since the extension of scalars

$$B(k) \to B(K)$$

merely multiplies each Hasse invariant by the local degree [31.9, MO]. On the other hand $S(\mathfrak{o})$ has been determined in at least one special case, namely when $k$ is a purely cyclotomic extension of $\mathbb{Q}$. See [R1]. Similar remarks apply to Theorem 2 below.

We wish to prove an analogue in the quadratic case as well. Suppose that $\omega$ is an involution (possibly the identity) on $\mathfrak{O}$ (and $K$). The *quadratic Brauer group* $B(\mathfrak{O}, \omega)$ is defined (see [H-T-W]) as the set of Morita equivalence classes of "$\omega$-antistructures" $(\Lambda, I, \varepsilon)$ where $\Lambda$ is an Azumaya algebra over $\mathfrak{O}$. (Thus $I$ is an $\omega$-antiautomorphism of $\Lambda$, $I^2$ is the inner automorphism with respect to the unit $\varepsilon$ of $\Lambda$, and $\varepsilon\varepsilon^I = 1$). There is a forgetful homomorphism

$$B(\mathfrak{O}, \omega) \xrightarrow{?} B(\mathfrak{O})$$

given by $[(\Lambda, I, \varepsilon)] \mapsto [\Lambda]$, whose image is known when $\mathfrak{O}$ is an unramified extension of the subring $\mathfrak{O}_0$ fixed by $\omega$ (see [S]): it consists of the elements of order 2 in $B(\mathfrak{O})$ if $\omega = id$, otherwise it is the kernel of the corestriction map $\mathrm{Cor}\colon B(\mathfrak{O}) \to B(\mathfrak{O}_0)$.

Now consider the *Schur subgroup* $S(\mathfrak{O}, \omega)$ of $B(\mathfrak{O}, \omega)$, which is defined as follows: If $G$ is any finite group, let $\Omega$ be the canonical $\omega$-involution on $KG$, i.e. the map which is $\omega$ on $K$ and inverts the elements of $G$. Then the Schur subgroup $S(\mathfrak{O}, \omega)$ of the quadratic Brauer group $B(\mathfrak{O}, \omega)$ consists of the Morita classes which contain an antistructure $(\Lambda, I, 1)$ (where $\Lambda$ is an Azumaya algebra over $\mathfrak{O}$) with the following property: There exists a finite group $G$ and a central simple direct summand $A$ of the group algebra $KG$ such that

(i) $A$ is stable under $\Omega$,
(ii) $\Lambda$ is the image of the projection of $\mathfrak{O}G$ in $A$,
(iii) $I = \Omega|_\Lambda$.

Note that $\Lambda$ is stable under $\Omega$ because $A$ and $\mathfrak{O}G$ both are.

Let $^*$ denote complex conjugation and define $\widetilde{K}$ to be the subfield of $K$ of elements fixed by the composition $^*\omega$. Let $\widetilde{\mathfrak{O}} = \mathfrak{O} \cap \widetilde{K}$.

**Theorem 2.** *Let $k$ be a subfield of $K$ containing $\widetilde{K}$ such that $k^\omega = k$, and let $\mathfrak{o} = \mathfrak{O} \cap k$. Then*

$$S(\mathfrak{O}, \omega) = \mathfrak{O} \otimes S(\mathfrak{o}, \omega).$$

*The forgetful map*

$$S(\widetilde{\mathfrak{O}}, \omega) \xrightarrow{?} S(\widetilde{\mathfrak{O}})$$

*is onto and*

$$\mathrm{im}(S(\mathfrak{O}, \omega) \xrightarrow{?} S(\mathfrak{O})) = \mathfrak{O} \otimes S(\widetilde{\mathfrak{O}}).$$

## The proofs

We begin with a simple lemma, which in fact holds in the more general case when $\mathfrak{O}$ is a Dedekind domain and $K/k$ is an arbitrary finite separable extension.

**Lemma.** *Let $G$ be a finite group, and suppose that the $k$-algebra $A'$ is a direct summand of $kG$. If the projection of $\mathfrak{O}G$ on $K \otimes A'$ is a maximal order, then the projection of $\mathfrak{o}G$ on $A'$ is also maximal.*

*Proof.* Suppose that $M'$ is an order in $A'$ which contains the projection $\Lambda'$ of $\mathfrak{o}G$. The projection of $\mathfrak{O}G$ on $K \otimes A'$ is the $\mathfrak{O}$-lattice $\mathfrak{O}\Lambda'$ generated by $\Lambda'$. And the $\mathfrak{O}$-lattice $\mathfrak{O}M'$ spanned by $M'$ is obviously an $\mathfrak{O}$-order which contains $\mathfrak{O}\Lambda'$, and so $= \mathfrak{O}\Lambda'$. It is well known that there is a basis $\{x_i\}$ of $A'$ and ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ in $\mathfrak{o}$ such that $M'$ is the direct sum of the $\mathfrak{a}_i x_i$ and $\Lambda'$ is the direct sum of the $\mathfrak{a}_i \mathfrak{b}_i x_i$. Since $\mathfrak{O}M' = \mathfrak{O}\Lambda'$, it follows that $\mathfrak{b}_1 = \mathfrak{b}_2 = \cdots = \mathfrak{o}$, and so $M' = \Lambda'$. □

We now begin the proof of Theorem 1. Suppose that $\Lambda$ is in $S(\mathfrak{O})$, arising, say, from the central simple direct summand $A$ of $KG$. Let $\chi$ be the absolutely irreducible character which belongs to $A$, and let $A'$ be the simple summand of $kG$ corresponding to $\chi$. Since the center of $A$ is $K = K(\chi)$ (Exercise 9.15, [I]), the values of $\chi$ lie in $K$, and so also in $K \cap \mathbb{Q}(\mu) \subseteq k$. Thus $A'$ is also central and $A = K \otimes A'$. It follows that $\Lambda = \mathfrak{O} \otimes \Lambda'$ where $\Lambda'$ is the projection of $\mathfrak{o}G$ on $A'$, and so by the above theroem $\Lambda'$ is also a maximal order. We claim that $\Lambda'$ is actually an Azumaya algebra. By 6.33 and 6.34 in [O-S], it suffices to prove it is split at every prime $\mathfrak{p}$ of $\mathfrak{o}$. Let $\mathfrak{P}$ be a prime of $\mathfrak{O}$ lying above $\mathfrak{p}$. Then

$$\Lambda_{\mathfrak{P}} = \mathfrak{O}_{\mathfrak{P}} \otimes \Lambda'_{\mathfrak{p}} \tag{1}$$

where the subscripts $\mathfrak{p}$ and $\mathfrak{P}$ denote completion. We must show that the fact that $\Lambda_{\mathfrak{P}}$ is isomorphic to a full matrix algebra over $\mathfrak{O}_{\mathfrak{P}}$ implies that $\Lambda'_{\mathfrak{p}}$ is isomorphic to one over $\mathfrak{o}_{\mathfrak{p}}$. Indeed were this not the case, the discriminant $d(\Lambda'_{\mathfrak{p}})$ would be properly contained in $\mathfrak{o}_{\mathfrak{p}}$ (see 20.3 and 20.4, [MO]). By (1) the discriminant of $\Lambda_{\mathfrak{P}}$ is $\mathfrak{O}_{\mathfrak{P}} d(\Lambda'_{\mathfrak{p}})$ which is properly contained in $\mathfrak{O}_{\mathfrak{P}}$. Contradiction. (This can also be deduced from the above lemma, and the theorem in [J]).

It follows that $\Lambda'$ is also an Azumaya algebra, and since

$$\Lambda = \mathfrak{O} \otimes \Lambda',$$

we get the desired result at once. □

We now turn to the proof of Theroem 2. Suppose that a Morita class in $S(\mathfrak{O}, \omega)$ contains the antistructure $(\Lambda, I, 1)$ arising from $\mathfrak{O}G$ as described earlier. Then $\Lambda$ is an order in the central simple $K$-algebra $A$ belonging to the absolutely irreducible character $\chi$, say. As in the proof of Theorem 1, the values of $\chi$ lie in $K_c$. But in fact we can show that the values lie in $\widetilde{K}$ as follows (cf. the proof of Lemma 1 in [R2]). The idempotent $e_\chi$ of $A$ is given by the formula

$$e_\chi = \frac{n}{g} \sum_{s \in G} \chi(s^{-1}) s$$

where $n = \chi(1)$ and $g$ is the order of $G$. Since $A$ is stable under $\Omega$, $e_\chi$ is fixed by $\Omega$, whence

$$\sum_{s \in G} \chi(s^{-1})^\omega s^{-1} = \sum_{s \in G} \chi(s^{-1}) s .$$

Using $\chi(s) = \chi(s^{-1})^*$ to transform the sum on the right side, we see that $\chi(s^{-1})^{*\omega} = \chi(s^{-1})$ for all $s$ in $G$, and so the values of $\chi$ do indeed lie in $\widetilde{K}$, and so also in $k$.

If $A'$ is the corresponding simple direct summand $A'$ of $kG$, it follows that it is also central. Let $\Lambda'$ be the projection of $\mathfrak{o}G$ on $A'$. By the same argument as in Theoem 1, $\Lambda'$ is an Azumaya algebra.

$A'$ is stable under $\Omega'$ since $A = K \otimes A'$. Since $\Lambda'$ is spanned over $\mathfrak{o}$ by the projections in $A'$ of the elements of $G$, it also is stable under $\Omega'$. If $I'$ is the restriction of $\Omega'$ to $\Lambda'$, we see that

$$(\Lambda, I) = \mathfrak{O} \otimes (\Lambda', I') .$$

This shows that $S(\mathfrak{O}, \omega) \subseteq \mathfrak{O} \otimes S(\mathfrak{o}, \omega)$, in fact that

$$S(\mathfrak{O}, \omega) = \mathfrak{O} \otimes S(\mathfrak{o}, \omega) . \tag{2}$$

Let $\widetilde{\Omega}$ be the canonical $\omega$-involution on $\widetilde{K}G$ and consider the pairing $\frac{1}{g} T(xy^{\widetilde{\Omega}})$ of $\widetilde{K}G$ into $\widetilde{K}$ where $T$ is the ordinary algebra trace. Since $\omega = {}^*$ on $\widetilde{K}$, it is easy to see that this pairing is an hermitian form with respect to complex conjugation. Moreover, $G$ forms an orthonormal basis and so the form is positive definite; thus $\frac{1}{g} T(xx^{\widetilde{\Omega}}) > 0$ for all $x \neq 0$ in $\widetilde{K}G$, and it follows at once that every simple direct summand of $\widetilde{K}G$ is stable under $\widetilde{\Omega}$. Therefore the projection of $\widetilde{\mathfrak{O}}G$ on such a summand is also stable, whence

$$S(\widetilde{\mathfrak{O}}, \omega) \to S(\widetilde{\mathfrak{O}})$$

is onto. We then obtain

$$\operatorname{im}(S(\mathfrak{O}, \omega) \to S(\mathfrak{O})) = \mathfrak{O} \otimes S(\widetilde{\mathfrak{O}})$$

by application of (2) in the case $k = \widetilde{K}$. □

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Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada