ZERO CYCLES ON QUADRIC HYPERSURFACES

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Abstract. Let $X$ be a projective quadric hypersurface over a field of characteristic not 2. It is shown that the Chow group $A_0(X)$ of 0-cycles modulo rational equivalence is infinite cyclic, generated by any point of minimal degree.

Let $k$ be a field of characteristic not 2 and let $X \subset \mathbb{P}^{d+1}_k$ be a quadric hypersurface defined by an equation $q = 0$ where $q$ is a quadratic form in $d + 2$ variables over $k$. In [4] I computed the $K$-theory of $X$ assuming that $q$ is nondegenerate. However the problem of computing the Chow groups of $X$, which was proposed in [3] is, to the best of my knowledge, still open. I will treat here the first nontrivial case by determining the Chow group $A_0(X)$ of 0-cycles modulo rational equivalence [1]. The result turns out to hold also in the singular case. My original proof made use of the results of [4]. I would like to thank Mohan Kumar for pointing out that this was not necessary and that only elementary facts about $A_0$ are needed.

Theorem. Let $X \subset \mathbb{P}^{d+1}_k$ be defined by $q = 0$ where $q$ is a quadratic form over a field $k$ of characteristic not 2 and $d > 0$. Then $A_0(X) = \mathbb{Z}$. It is generated by any rational point if one exists. If $X$ has no rational point then $A_0(X)$ is generated by any point of degree 2 over $k$.

We can obviously assume that $q$ is not identically 0 so that $\dim X = d$. If $d = 0$, it is then clear that $A_0(X) = \mathbb{Z} \times \mathbb{Z}$ if $X$ consists of two rational points and $A_0(X) = \mathbb{Z}$ otherwise. For the reader's convenience I will restate the following standard classification for the case $d = 1$.

Lemma 1. Let $X \subset \mathbb{P}^2_k$ be a quadric curve. Then one of the following holds:

1. $X = \mathbb{P}^1$ embedded in $\mathbb{P}^2$ by the Veronese embedding.
2. $X$ is a smooth conic with no rational point.
3. $X = L_1 \cup L_2$ is a union of two lines defined over $k$ with one common point.

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(4) $X = L \cup L'$ where $L$ is a line defined over a quadratic extension of $k$ (but not over $k$) and $L'$ is its conjugate. The only rational point on $X$ is the intersection $L \cap L'$.

(5) $X$ is a double line defined by $l^2 = 0$ where $l$ is a linear form over $k$.

If $q$ is nondegenerate and nonisotropic we have case (2). In the isotropic case we can write $q = xy - z^2$, getting case (1). If $q$ reduces to $ax^2 + by^2$ with $ab \neq 0$, we have case (3) if $-a^{-1}b$ is a square in $k$, and case (4) otherwise. If $q$ reduces to $ax^2$ we have case (5).

I will write $x \sim y$ when $x$ and $y$ are rationally equivalent. In the following lemmas, $X$ will always denote the quadric hypersurface of the theorem.

**Lemma 2.** If $x$ and $y$ are rational points of $X$ then $x \sim y$.

**Proof.** Let $P$ be a 2-plane in $P^{d+1}$ containing the points $x$ and $y$. Then $P \cong P^2$. If $P \subset X$ the result is clear. Otherwise $X \cap P$ is as in Lemma 1. In case (1) the result is again clear. Case (2) cannot occur. In case (3) $x$ and $y$ are rationally equivalent to the common point $a = L_1 \cap L_2$. In case (4) there is only one rational point, so $x = y$. Finally, in case (5) $x$ and $y$ lie on the line $l = 0$, so $x \sim y$.

**Lemma 3.** The theorem is true if $X$ has a rational point.

**Proof.** Let $x$ be a rational point and let $y$ be any point. Let $k' = k(y)$ be the residue field of $y$ and let $X' = k' \otimes_k X$ with projection $\pi: X' \to X$. Then $X'$ is just the quadric hypersurface defined by $q = 0$ over $k'$. Now $\pi^{-1}(y) = \text{Spec} \ k' \otimes_k k(y)$ has a rational point $y'$ and $\pi^{-1}(x) = \text{Spec} \ k' \otimes_k k(x) = \text{Spec} \ k' = \{x',x''\}$ with $x'$ rational. By Lemma 2, $x' \sim y'$. Therefore $y = \pi_1(x') \sim \pi_1(x') = |k':k|x$. Thus $A_0(X)$ is generated by $x$. Since $\deg: A_0(X) \to \mathbb{Z}$ by $\deg z = |k(z):k|$, the result follows.

We can now assume that $X$ has no rational point. Since we can write $q = \sum a_i x_i^2$, it is clear that $X$ has points of degree 2. Also $X$ must be smooth since all $a_i$ must be nonzero. The following is a special case of [3, Lemma 13.4].

**Lemma 4.** If $X$ has no rational point the all points of $X$ have even degree.

**Proof.** Suppose $x \in X$ has odd degree. Let $k' = k(x)$. Then $k' \otimes_k X$ has a rational point so that $q$ is isotropic over $k'$. Since $|k':k|$ is odd, a theorem of Springer [2, Chap. 7, Theorem 2.3] implies that $q$ is isotropic over $k$, so $X$ has a rational point.

**Lemma 5.** Let $K$ be the kernel of $\deg: A_0(X) \to \mathbb{Z}$. Then $2K = 0$.

**Proof.** We can assume that $X$ has no rational point. Let $x \in X$ have degree 2 and set $k' = k(x)$. Let $X' = k' \otimes_k X$ with projection $\pi: X' \to X$. Then $\pi^{-1}(\cdot) = \text{Spec} k' \otimes_k k(x) = \{x',x''\}$ where $x'$ and $x''$ are rational over $k'$. If $y$ is any closed point, $\pi^{-1}(y) = \text{Spec} k' \otimes_k k(y) = \{y',y''\}$ or $\{z\}$.
depending on whether \( k' \otimes_k \kappa(y) \) splits or not. By Lemma 3 we can write \( y' \sim mx' \) or \( z \sim mx' \) getting either \( y = \pi_*(y') \sim m\pi_*(x') = mx \) or \( 2y = \pi_*(z) \sim mx \). It follows that twice any 0-cycle is rationally equivalent to a multiple of \( x \).

**Lemma 6.** If \( x \) and \( y \) have degree 2 then \( x \sim y \).

**Proof.** We can assume \( X \) has no rational point by Lemma 3. Let \( V = X(\overline{k}) \) be the variety corresponding to \( X \) over the algebraic closure \( \overline{k} \) of \( k \). The point \( x \) corresponds to a pair of points \( \xi, \xi' \) of \( V \). Since \( \text{char} k \neq 2 \), \( \kappa(x) \) is Galois over \( k \) so that \( \xi \) and \( \xi' \) are distinct and conjugate over \( k \). The line \( L \) spanned by \( \xi \) and \( \xi' \) is stable under the Galois group and therefore is defined over \( k \). Since \( (L \cdot X) = 2 \) by Bezout's theorem, we see that \( L \cdot X = x \).

Similarly, \( y = L' \cdot X \) for some line \( L' \) defined over \( k \). Since \( L \sim L' \) as 1-cycles on \( P^{d+1} \), it follows that \( x \sim y \).

If \( y \) is a closed point of \( X \) I will say that \( y \) is "good" if \( y \sim mx \) for some integer \( m \) and some point \( x \) of degree 1 or 2. The theorem will follow from Lemmas 3 and 6 if we can show that all points are good.

**Lemma 7.** Let \( n: X' = k' \otimes_k X \to X \) be the canonical projection where \( |k':k| \) is odd. If all points of \( n^{-1}(y) \) are good, so is \( y \).

**Proof.** We can assume that \( X \) has no rational points. The same is then true of \( X' \) by Springer's theorem as in the proof of Lemma 4. Therefore if \( x \) is a point of degree 2 on \( X \) then \( n^{-1}(x) = \{x'\} \), since otherwise \( n^{-1}(x) \) would consist of two rational points.

Let \( k' \otimes_k \kappa(y) = A_i \times \cdots \times A_i \) where the \( A_i \) are local artinian with residue fields \( k_i = A_i/\mathfrak{M}_i = \kappa(y_i) \) where the \( y_i \) are the points of \( n^{-1}(y) \). Then \( |k':k| = \dim_k \kappa(y) = \sum |A_i| |k_i:k(y)| \) so for some \( i \), \( |\kappa(y_i):k(y)| \) is odd. Since \( y_i \) is good, \( y_i \sim mx' \) for some \( m \) and hence \( \pi_*(y_i) = |\kappa(y_i):k(y)| y \sim m\pi_*(x') = m|k_i:k(x)|. \) Since \( |\kappa(y_i):k(y)| \) is odd and \( 2y \sim mx \) for some \( m \) by Lemma 5, the result follows.

**Lemma 8.** Let \( \eta: X' = k' \otimes_k X \to X \) be the canonical projection where \( |k':k| = 2 \). Let \( y \) be a closed point of \( X \) such that \( \eta^{-1}(y) = \{y', y''\} \) has two distinct points. If \( y' \) is good, so is \( y \).

**Proof.** We can assume that \( X \) has no rational points. If \( x' \in X' \) is rational, then \( x = \eta_*(x') \) has degree 2 and \( y' \sim mx' \) implies \( y = \eta_*(y') \sim mx \). If \( X' \) has no rational point and \( x \in X \) has degree 2, then \( \eta^{-1}(x) = \{x'\} \) where \( x' \) has degree 2 on \( X' \). Now \( y' \sim mx' \), so \( y = \eta_*(y') \sim m\eta_*(x') = 2mx \) as required.

We will now show that all closed points of \( X \) are good. Let \( y \in X \) have degree \( n \). By induction, we can assume that all points of degree less than \( n \) are good on all the hypersurfaces \( k' \otimes_k X \). Let \( K \) be the "normal closure" of \( \kappa(y) \), i.e., the composite of all conjugates of \( \kappa(y) \) in the algebraic closure
$\bar{k}$ of $k$. Let $G = \text{Aut}(K/k)$. Then $K^G$ is purely inseparable over $k$ and therefore of odd degree since $\text{char} \ k \neq 2$. Let $H$ be a 2-Sylow subgroup of $G$. Then $k' = K^H$ is of odd degree over $K^G$ and hence over $k$. Let $\pi: X' = k' \otimes_k X \to X$ be the canonical projection. By Lemma 7 it will suffice to show that each point $y'$ of $\pi^{-1}(y)$ is good. Now $\kappa(y')$ is a quotient of $k' \otimes_k \kappa(y)$. Any embedding of $\kappa(y')$ in $\bar{k}$ fixing $k'$ will necessarily send $\kappa(y)$ into $K$ so we get $k' = K^H \subset \kappa(y') \subset K$ and $\kappa(y') = K^L$ for some subgroup $L$ of $H$. If $L = H$, then $y'$ is rational and hence good. If $L < H$, let $M$ be a maximal proper subgroup of $H$ containing $L$. Then $|H:M| = 2$. Let $k'' = K^M$ and consider $\eta: X'' = k'' \otimes_k X \to X'$. Since $k'' = K^M \subset K^L = \kappa(y')$ and $|k'':k'| = 2$, $k'' \otimes_k, \kappa(y') \approx \kappa(y') \times \kappa(y')$ so $\eta^{-1}(y') = \{u,v\}$ where $\kappa(u) = \kappa(y'(y') = \kappa(v)$. Since $|\kappa(y'):k''| = \frac{1}{2}|\kappa(y'):k'|$, the induction hypothesis shows that $u$ is good. Therefore $y'$ is good by Lemma 8 and hence $y$ is good by Lemma 7.

The corresponding result for the affine case now follows easily.

**Corollary.** Let $V \subset \mathbb{A}^{d+1}_k$ be the affine hypersurface defined by $q = 1$ where $q$ is a quadratic form over a field $k$ of characteristic not 2 and $d > 0$. If $q$ is nonisotropic and represents 1 then $A_0(V) = \mathbb{Z}/2\mathbb{Z}$. In all other cases $A_0(V) = 0$.

**Proof.** Let $X \subset \mathbb{P}^{d+1}_k$ be defined by $q - y^2 = 0$ and let $X_\infty = X \cap (y = 0)$. Then $V = X - X_\infty$ and we have an exact sequence $A_0(X_\infty) \to A_0(X) \to A_0(V) \to 0$. Since $\deg: A_0(X) \to \mathbb{Z}$ is injective by the theorem, it follows that $A_0(V) = \deg A_0(X)/\deg A_0(X_\infty)$, which is zero unless $X$ has a rational point and $X_\infty$ does not, in which case it is $\mathbb{Z}/2\mathbb{Z}$.

**References**


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