UNIQUENESS OF APERIODIC KNEADING SEQUENCES

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Abstract. The trapezoidal function \( f_e(x) \) is defined for fixed \( e \in (0, 1/2) \) by \( f_e(x) = (1/e)x \) for \( x \in [0, e] \), \( f_e(x) = 1 \) for \( x \in (e, 1-e) \), and \( f_e(x) = (1/e)(1-x) \) for \( x \in [1-e, 1] \). For a given \( e \) and the associated one-parameter family of maps \( \{k \cdot f_e(x) \mid k \in [0, 1]\} \), we show that if \( A \) is an aperiodic kneading sequence, then there is a unique \( A \in [0, 1] \) so that the itinerary of \( A \) under the map \( k \cdot f_e \) is \( A \). From this, we conclude that the "stable windows" are dense in \([0, 1]\) for the one-parameter family \( k \cdot f_e \).

This note is mainly concerned with those maps which are trapezoidal. The trapezoidal function, \( f_e \), is defined for \( e \in (0, 1/2) \) by \( f_e(x) = x/e \) for \( x \in [0, e] \), \( f_e(x) = 1 \) for \( x \in (e, 1-e) \), and \( f_e(x) = (1-x)/e \) for \( x \in [1-e, 1] \). For a fixed \( e \), one can form a one parameter family of maps by considering the set \( \{k \cdot f_e(x) \mid k \in [0, 1]\} \).

Throughout this note, the notation and terminology of Beyer, Mauldin, and Stein (BMS) [2] is used. Thus, if \( g \) maps \([0, 1]\) into itself and \( x \in [0, 1] \), then the itinerary of \( x \) under \( g \) is given by \( I^g(x) = \{b_i\}_{i \geq 0} \), where \( b_i = R \) if \( g^i(x) > 1/2 \), \( b_i = L \) if \( g^i(x) < 1/2 \), and \( b_i = C \) if \( g^i(x) = 1/2 \) (if \( b_i = C \) for some \( i \), then the itinerary stops). We note that \( I^g(x) \) is either an infinite sequence of \( R \)'s and \( L \)'s or is a finite sequence of \( R \)'s and \( L \)'s followed by a \( C \). If \( g \) is unimodal and \( \lambda \in [0, 1] \) is such that the orbit of \( \lambda \) under the scaled map \( \lambda g \) contains \( 1/2 \), then the finite sequence \( I^g(\lambda) \) is referred to as an MSS sequence [2, 9].

For \( g \) unimodal and \( \lambda \in [0, 1] \), the itinerary of \( \lambda \) under the map \( \lambda g \), \( I^g(\lambda) \), is referred to as the kneading sequence of \( \lambda g \) [6]. BMS show that \( I^g(\lambda) \) is shift maximal in the parity-lexicographical order when \( g \) is unimodal and \( \lambda \in [0, 1] \) (throughout this note, when comparing kneading sequences the parity-lexicographical order is used). Furthermore, if \( B \) is a finite or infinite shift maximal sequence, then there is some unimodal map \( g \) and some \( \lambda \in [0, 1] \) so that \( I^g(\lambda) = B \). Thus any kneading sequence is shift maximal, and any shift maximal sequence is a kneading sequence. We note that if \( g \) is unimodal and...
\( \lambda \in [0, 1] \), then \( I^g(\lambda) \) is either an MSS sequence, or is infinite and periodic, or is infinite and aperiodic. In this note we prove the following theorem.

**Theorem A.** Fix \( e \in (0, 1/2) \). Let \( A \) be aperiodic shift maximal sequence. Then there exists a unique \( \lambda \in [0, 1] \) such that \( I^{k_e}(\lambda) = A \).

Let us comment on our interest in Theorem A in general terms. For a unimodal map \( g \) let

\[ \mathcal{P}_g = \{ \lambda \in [0, 1] \mid I^g(\lambda) \text{ is an MSS sequence or is infinite and periodic}\}. \]

It is widely believed [7; 4, pp. 31, 69] that \( \mathcal{P}_{4x(1-x)} \) is dense in \([0, 1]\). We note that \( \mathcal{P}_{4x(1-x)} \) consists of precisely those \( \lambda \) such that \( 4x(1-x) \) has a stable orbit [4, p. 69]. Similarly, \( \mathcal{P}_{f_e} \) consists of those values of \( \lambda \) such that \( \lambda f_e \) has a stable orbit or a finite orbit containing either \( e \) or \( 1 - e \). Theorem A and Lemma 1.1 (of Appendix A) imply that \( \mathcal{P}_{f_e} \) is dense in \([0, 1]\), i.e., the “stable windows” are dense in \([0, 1]\). If one could prove Theorem A for the family \( \lambda 4x(1-x) \), then again the “stable windows” would be dense in \([0, 1]\). Note that there are uncountably many aperiodic shift maximal sequences.

We say that the one-parameter family \( \{ \lambda g \mid \lambda \in [0, 1] \} \), where \( g \) is unimodal, exhibits uniqueness provided that for each MSS sequence \( P \) there exists exactly one \( \lambda \) such that \( I^{\lambda g}(\lambda) = P \). Moreover, the family \( \{ \lambda g \mid \lambda \in [0, 1] \} \) is said to be fully unique if it exhibits uniqueness and if for each aperiodic kneading sequence \( A \) there is exactly one \( \lambda \) such that \( I^{\lambda g}(\lambda) = A \). Fix \( e \in (0, 1/2) \). It is known that \( \{ \lambda f_e \mid \lambda \in [0, 1] \} \) and \( \{ \lambda 4x(1-x) \mid \lambda \in [0, 1] \} \) exhibit uniqueness [2, 8, 10]. In this note, we establish that the family \( \{ \lambda f_e \} \) is fully unique. We remark that this is the only one-parameter family shown to be fully unique. If we take \( g(x) \) to be \( 4x(1-x) \), then for certain but not all aperiodic kneading sequences Dennis Sullivan has shown that there is a unique \( \lambda \) so that \( I^{\lambda g}(\lambda) \) is the given kneading sequence [11]. Moreover he has shown that if there exists an analytic family that is fully unique, then the family \( \lambda 4x(1-x) \) is fully unique [11].

This paper is broken into two sections. Section one consists of general observations, and preliminary comments. Section two contains the proof of Theorem A.

BMS [2] show that the one-parameter family \( \lambda f_e \) exhibits uniqueness for \( 0 < e < (3\sqrt{17} - 11)/4 \) (we note that \( (3\sqrt{17} - 11)/4 \equiv 0.3423 \)). Metropolis and Louck [8] show uniqueness for any \( e \in (0, 1/2) \). Our proof of Theorem A uses uniqueness for a given \( e \). If \( e \in (0, 1/2) \) and \( B \) is some periodic shift maximal sequence, it is known that there is more than one \( \lambda \) with \( I^{k_e}(\lambda) = B \). This will be discussed presently. Throughout the rest of this note assume that \( e \in (0, 1/2) \) is fixed. We state the following fact as a theorem and outline its proof in Appendix A.
Theorem B. Let $g$ be a unimodal Lipschitz continuous concave function that has a continuous derivative in a neighborhood of $x = 1/2$. Furthermore assume that $g$ exhibits uniqueness. Then the following hold.

(i) If $B$ is an aperiodic shift maximal sequence, then $\{\lambda \in [0,1]|I^g(\lambda) = B\}$ is a closed interval or consists of a single point.

(ii) If $B = \{b_j\}_{j \geq 1}$ is a periodic shift maximal sequence of period $k$, then exactly one of the following hold.

(a) The sequence $b_1 \cdots b_{k-1} C$ is the harmonic of some MSS sequence $P$, and the set of $\lambda$ for which $I^g(\lambda) = B$ is an open interval.

(b) The sequence $b_1 \cdots b_{k-1} C$ is not the harmonic of some MSS sequence, and the set of $\lambda$ for which $I^g(\lambda) = B$ is either an open or half open interval. More precisely, if $b_1 \cdots b_{k-1}$ is odd (even), then $(b_1 \cdots b_{k-1} R)^\infty$ ($(b_1 \cdots b_{k-1} L)^\infty$) is a left closed right open interval, and $(b_1 \cdots b_{k-1} L)^\infty$ $\cdot((b_1 \cdots b_{k-1} R)^\infty)$ is an open interval.

Recall that $e \in (0,1/2)$ is fixed. Set $f = f_\epsilon$. For $x \in \mathbb{R}$ and $\lambda \in (0,1]$ set

$$f^{-1}_{\lambda,R}(x) = 1 - (e/\lambda)x,$$ and

$$f^{-1}_{\lambda,L}(x) = (e/\lambda)x.$$

For $P = P_1 \cdots P_n \in \{R,L\}^n$ set

(i) $\rho(P) = \{j|P_j = R\}$, and

(ii) $G_{\lambda}(P, y) = f^{-1}_{\lambda,P_1}(f^{-1}_{\lambda,P_2}((f^{-1}_{\lambda,P_3}(y))\cdots))$.

Beyer and Ebanks [1] have the following theorem, which can be proved by induction on $n$.

Theorem 2.1. Let $P = P_1 \cdots P_n$ be a finite sequence of $R$'s and $L$'s. Then,

$$G_{\lambda}(P, y) = \sum_{j \in \rho(P)} (-1)^{1+|\rho(P)|} 1 - (e/\lambda)^j - 1$$

Remark 2.2. Let $A = \{a_i\}_{i \geq 1}$ be an aperiodic shift maximal sequence, and for each $n \in \mathbb{N}$ let $A|_n = (a_1, \ldots, a_n)$. Suppose that $I^{\lambda f}(\lambda) = A$. Observe the following two facts.

(i) For each $n \in \mathbb{N}$, $(\lambda f)^n(\lambda) \notin [e,1-e]$, for otherwise $A$ would be periodic.

(ii) Fix $n \in \mathbb{N}$. Then $G_{\lambda}(A|_n, (\lambda f)^n(\lambda)) = \lambda$. To see this note that $\lambda \rightarrow (\lambda f)(\lambda) = \lambda f_{a_1}(\lambda) \rightarrow (\lambda f)^2(\lambda) = \lambda f_{a_2}(\lambda)\cdots \rightarrow (\lambda f)^n(\lambda) = \lambda f_{a_n}(\lambda)\cdots(\lambda f_{a_{n-1}}(\lambda))\cdots)$, where $f_L(x) = (1/e)x$ and $f_R(x) = (1/e)(1-x)$, with $(\lambda f)^j(\lambda) \notin [e,1-e]$ for $1 \leq j \leq n$. 
Remark 2.3. For $P = RL^n$, let $\lambda_n$ be the unique scalar such that $I^{\lambda_n f}(\lambda_n) = P$. BMS show that $\lambda_n \to 1$ as $n \to \infty$. Thus Theorem A holds if $A$ is $RL^\infty$.

Proof of Theorem A. Recall that $A$ is an aperiodic shift maximal sequence. We assume, using Remark 2.3, that $A \neq RL^\infty$. Suppose there is more than one $\lambda$ such that $I^{\lambda f}(\lambda) = A$. Then Theorem B implies that there exists a closed interval $[\alpha, \beta] \subset [0,1]$ such that for every $\lambda \in [\alpha, \beta]$ we have $I^{\lambda f}(\lambda) = A$. Note that $\alpha \geq (1 - e)$. Again, say $A = \{a_i\}_{i \geq 1}$ and for each $n \in \mathbb{N}$ let $A_n$ denote the finite sequence $(a_1, \ldots, a_n)$.

Let $\lambda \in [\alpha, \beta]$ and $n \in \mathbb{N}$. Then, as in Remark 2.2,

$$(1) \quad G_\lambda(A_n, (\lambda f)^n(\lambda)) = \lambda.$$ 

Thus for every $\lambda \in [\alpha, \beta]$ and $n \in \mathbb{N}$ we have that (1) holds.

For $\lambda \in (e, 1]$, $n \in \mathbb{N}$ set

$$g_{\lambda,n}(x) = G_\lambda(A_n, (\lambda f)^n(x)), \quad x \in [0, 1].$$

Then,

$$g_{\lambda,n}(\lambda) = \lambda \quad \text{for all } \lambda \in [\alpha, \beta] \text{ and } n \in \mathbb{N}.$$ 

Also, for each $\lambda \in (e, 1]$ set

$$g(\lambda) = \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)| - 1} (e/\lambda)^{j-1}.$$ 

Notice that $g(\lambda)$ is the limit of the $g_{\lambda,n}(\lambda)$ as $n \to \infty$. We have $g(\lambda)$ and $g_{\lambda,n}(x)$ defined for $\lambda \in (e, 1]$, $n \in \mathbb{N}$, and $x \in [0, 1]$.

Next, fix $\lambda \in (e, 1]$, $n \in \mathbb{N}$, and $x \in [0, 1]$. Then,

$$g(\lambda) - g_{\lambda,n}(x) = \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)| - 1} (e/\lambda)^{j-1}$$

$$- (-1)^{|\rho(A_n)|} (e/\lambda)^n (\lambda f)^n(x).$$

Thus,

$$|g(\lambda) - g_{\lambda,n}(x)| \leq \sum_{j \in \rho(A)} ((e/\lambda)^{j-1}) + (e/\lambda)^n \lambda,$$

for $\lambda \in (e, 1]$, $n \in \mathbb{N}$, and $x \in [0, 1]$.

Choose $\gamma < 1$ so that $0 < (e/\lambda) < \gamma$ for $\lambda \in [\alpha, \beta]$. Then for $\lambda \in [\alpha, \beta]$, $n \in \mathbb{N}$, and $x \in [0, 1]$ we have

$$|g(\lambda) - g_{\lambda,n}(x)| \leq \sum_{j \in \rho(A)} (\gamma^{j-1}) + \gamma^n \beta.$$ 

Hence, for every $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$|g(\lambda) - g_{\lambda,m}(x)| < \epsilon$$

for all $\lambda \in [\alpha, \beta]$ and $x \in [0, 1]$. 

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In particular, for every \( \varepsilon > 0 \) there exists \( m \in \mathbb{N} \) so that
\[
|g(\lambda) - \lambda| = |g(\lambda) - g_{\lambda, m}(\lambda)| < \varepsilon
\]
for every \( \lambda \in [\alpha, \beta] \). Thus,
\[
g(\lambda) = \lambda
\]
on \([\alpha, \beta]\).

However recall that
\[
g(W_{\lambda}) = 2^{i - \lambda p_{\lambda}} p_{\lambda} - 1 \left( 1 - x_{i} / x_{j} \right),
\]
with \( p(\lambda) \) infinite. Setting \( t = (\varepsilon / \lambda) \) we find that
\[
\left( \sum_{j \in p(\lambda)} (-1)^{|\rho(a_{1}, \ldots, a_{j})|} (\varepsilon / \lambda)^{j-1} \right) - e = 0
\]
on \([\varepsilon / \beta, \varepsilon / \alpha]\). This is a contradiction. Thus we have proven Theorem A.

APPENDIX A

The following lemmas and theorems are used to prove Theorem B.

**Lemma 1.1.** Let \( f \) be unimodal, \( 0 < \lambda_1 < \lambda_2 < 1 \), and \( I^{\lambda_1 f}(\lambda_1) \), \( I^{\lambda_2 f}(\lambda_2) \) be distinct elements of \( \{R, L\}^{\mathbb{N}} \). Then there is some \( \lambda_0 \in (\lambda_1, \lambda_2) \) such that \( I^{\lambda_0 f}(\lambda_0) \) is finite.

**Proof.** Let \( k \) be the first index where \( I^{\lambda_1 f}(\lambda_1) \) and \( I^{\lambda_2 f}(\lambda_2) \) differ. Set
\[
\gamma = \sup\{\lambda \in (\lambda_1, \lambda_2) | I^{\lambda f}(\lambda) \text{ agrees with } I^{\lambda_1 f}(\lambda_1) \text{ in the first } k \text{ positions}\}.
\]
The \( \lambda_1 < \gamma < \lambda_2 \), and \( I^{\gamma f}(\gamma) \) is finite, since otherwise the definition of \( \gamma \) is contradicted.

The following two theorems are taken from BMS.

**Theorem 1.2.** Let \( f \) be a unimodal Lipschitz continuous function that has a continuous derivative in a neighborhood of \( x = 1/2 \). Suppose \( 0 < \lambda_1 < \lambda_2 < 1 \) and \( A \) is a shift maximal sequence other than \( L^\infty, C, R^\infty, \) or \( RL^\infty \). Suppose further that
\[
I^{\lambda_1 f}(\lambda_1) < A < I^{\lambda_2 f}(\lambda_2).
\]
Then there exists some \( \lambda \in (\lambda_1, \lambda_2) \) so that \( I^{\lambda f}(\lambda) = A \). (The theorem also holds if \( I^{\lambda_1 f}(\lambda_1) > A > I^{\lambda_2 f}(\lambda_2) \).

**Theorem 1.3.** Let \( f \) be a unimodal Lipschitz continuous concave function with a continuous derivative in a neighborhood of \( 1/2 \). Then for each shift maximal sequence \( P \) there is a value of \( \lambda \) so that \( I^{\lambda f}(\lambda) = P \).

In the proof of Theorem 1.2 BMS prove the following lemma.
Lemma 1.4. Let $f$ be a unimodal Lipschitz continuous function that has a continuous derivative in a neighborhood of $x = 1/2$. Suppose $\lambda_0 \in [0, 1]$ is such that $I^{\lambda_0 f}(\lambda_0)$ is finite. Say, $I^{\lambda_0 f}(\lambda_0) = PC$, where $P \in \{R, L\}^n$ for some $n$. Then there exists an open interval $U \subset [0, 1]$ containing $\lambda_0$ so that if $\lambda$ is in $U$, then

$$I^{\lambda f}(\lambda) \in \{PC, (PR)^\infty, (PL)^\infty\}.$$  

Proof of Theorem B (i). We note that, by Theorem 1.3, there is some $\lambda$ with $I^{\lambda g}(\lambda) = B$. Let

$$\beta = \sup\{\lambda \in [0, 1] | I^{\lambda g}(\lambda) = B\},$$

and

$$\alpha = \inf\{\lambda \in [0, 1] | I^{\lambda g}(\lambda) = B\}.$$ 

Suppose that $\alpha < \beta$. Then Lemma 1.4 implies that both $I^{\alpha g}(\alpha)$ and $I^{\beta g}(\beta)$ are not finite and therefore must both be equal to $B$. If $I^{\lambda g}(\lambda) = B$ for all $\lambda \in [\alpha, \beta]$, then we are done. Suppose there is some $\lambda \in [\alpha, \beta]$ such that $I^{\lambda g}(\lambda) \neq B$. Then, by Lemma 1.1, there exists some $\lambda_0 \in [\alpha, \beta]$ such that $I^{\lambda_0 g}(\lambda_0) = Q$ is finite.

Case 1. Suppose $Q > B$. Then, using Theorem 1.2, there is some $\lambda > \beta$ so that $Q = I^{\lambda g}(\lambda)$. This contradicts uniqueness.

Case 2. Suppose $Q < B$. The argument is similar to case one.

Remark 1.5. We briefly recall what the harmonics of an MSS sequence are. Let $P \in \{R, L\}^k$ be an MSS sequence; we have temporarily dropped the $C$. For $n \in \mathbb{N}$, set

$$H_n(P) = \begin{cases} H_{n-1}(P)LH_{n-1}(P), & \text{if } H_{n-1}(P) \text{ is odd,} \\ H_{n-1}(P)RH_{n-1}(P), & \text{if } H_{n-1}(P) \text{ is even,} \end{cases}$$

where $H_0(P) = P$.

Then $H_n(P)$ is called the $n$th harmonic of $P$. We let $H_\infty(P)$ be the unique element in $\{R, L\}^\mathbb{N}$ that is the common extension of the harmonics of $P$ and note that $H_\infty(P)$ is an aperiodic kneading sequence. Of course, $H_m(P)$ is odd (even) if there is an odd (even) number of $R$'s in $H_m(P)$. The following facts are known [5, 9, 3] (for (ii) see [3, Theorem 3.2, p. 436]).

(i) If $P$ is an MSS sequence and $Q$ is some shift maximal sequence so that $P < Q < H_1(P)$, then $Q = (PR)^\infty$ if $H_1(P) = PRP$ or $Q = (PL)^\infty$ if $H_1(P) = PLP$.

(ii) If $B = \{b_i\}_{i \geq 1}$ is a periodic shift maximal sequence of period $k$, then $b_1 \cdots b_{k-1}C$ is an MSS sequence. Moreover, if $Q \neq b_1 \cdots b_{k-1}$ is some MSS sequence with $a \in \{R, L\}$ so that $(Qa)^\infty = B$, then $Q$ is the first harmonic of $b_1 \cdots b_{k-1}C$.

Proof of Theorem B (ii). Theorem B (ii) now follows from Remark 1.5, Lemma 1.4, and Theorem 1.2.
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