COMPLEMENTED COPIES OF \( c_0 \)
IN VECTOR-VALUED HARDY SPACES

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Abstract. Let \( X \) be a complex Banach space containing a copy of \( c_0 \), let \( T \) be the unit circle and let \( D \) be the open unit disk in the complex plane. Then \( H^p(T, X) \) contains a complemented copy of \( c_0 \) for \( 1 \leq p < \infty \). The corresponding result for \( H^p(D, X) \) fails for \( 1 \leq p \leq \infty \).

1. Introduction

If \( X \) is a Banach space which contains a copy of \( c_0 \) then \( L^p([0,1], X) \) contains a complemented copy of \( c_0 \) for \( 1 \leq p < \infty \) [5]. In this note we consider the corresponding problem for vector-valued Hardy spaces. However, there are two natural Hardy spaces to consider, \( H^p(T, X) \) and \( H^p(D, X) \). We will show that \( H^p(T, X) \) contains a complemented copy of \( c_0 \) whenever \( 1 \leq p < \infty \) and \( X \) is a complex Banach space containing a copy of \( c_0 \). The proof will allow us to extend the result to a slightly larger class of spaces. We will also show that the spaces \( H^p(D, L^\infty) \) do not contain complemented copies of \( c_0 \) for \( 1 \leq p \leq \infty \).

2. Preliminaries and results

Throughout this note \( T \) will denote the unit circle, \( \frac{d\theta}{2\pi} \) will denote normalized Lebesgue measure on \( T \), and \( D \) will be the open unit disk in the complex plane.

Let \( X \) be a complex Banach space and let \( 1 \leq p \leq \infty \). The space \( H^p(D, X) \) is the collection of all \( X \)-valued analytic functions on \( D \) with \( \|f\|_p < \infty \) where

\[
\|f\|_p = \sup_{0 \leq r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} \right\}^{1/p}
\]

for \( 1 \leq p < \infty \), and

\[
\|f\|_\infty = \sup_{z \in D} \|f(z)\|.
\]
If \( f : T \to X \) is a vector-valued function then its Fourier coefficients are

\[
\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}, \quad \text{for each } n \in \mathbb{Z}.
\]

For \( 1 \leq p \leq \infty \), we define

\[
H^p(T, X) = \{ f \in L^p(T, X) : \hat{f}(n) = 0 \quad \text{for all } n < 0 \}.
\]

Before we get to the main result we need a lemma which appears implicitly in [2] and [7].

**Lemma.** If a Banach space \( X \) contains a sequence \( (x_n)_{n=1}^\infty \) which is equivalent to the unit vector basis of \( c_0 \) and if \( (x_n^*)_{n=1}^\infty \) is a weak* null sequence in \( X^* \) such that \( \inf_n |x_n^*(x_n)| > 0 \), then \( X \) contains a complemented copy of \( c_0 \).

**Proof.** Define an operator \( S : X \to c_0 \) by \( S(x) = (x_n^*(x))^\infty_{n=1} \). Clearly, \( S \) is well defined, since \( (x_n^*)_{n=1}^\infty \) is weak* null, and also bounded and linear. The series \( \sum_{n=1}^\infty x_n \) is weakly unconditionally Cauchy but \( \sum_{n=1}^\infty S(x_n) \) is not unconditionally convergent in \( c_0 \) because \( \inf_n |x_n^*(x_n)| > 0 \). By [1] there is a subsequence \( (y_n)_{n=1}^\infty \) of \( (x_n)_{n=1}^\infty \) such that \( (y_n)_{n=1}^\infty \) is equivalent to the unit vector basis of \( c_0 \) and \( S|_{[y_n]_{n=1}^\infty} \) is an isomorphism of \( [y_n]_{n=1}^\infty \) onto \( Y = [S(y_n)]_{n=1}^\infty \). \( Y \) is a subspace of \( c_0 \) which is isomorphic to \( c_0 \) and so is complemented in \( c_0 \) by a bounded linear projection \( Q \) (see [8]). Consider the operator \( P : X \to X \) defined by \( P(x) = (S|_{[y_n]_{n=1}^\infty})^{-1}QS(x) \) for \( x \in X \). \( P \) is a bounded linear projection of \( X \) onto \( [y_n]_{n=1}^\infty \), and since \( [y_n]_{n=1}^\infty \) is isomorphic to \( c_0 \), the proof is complete.

**Theorem.** Let \( X \) be a complex Banach space and \( 1 \leq p < \infty \). If \( X \) contains a copy of \( c_0 \), then \( H^p(T, X) \) contains a complemented copy of \( c_0 \).

**Proof.** Let \( (x_n)_{n=1}^\infty \) be a sequence in \( X \) equivalent to the unit vector basis of \( c_0 \). Then there are constants \( K_1, K_2 > 0 \) so that for any choice of scalars \( a_1, a_2, \ldots, a_n \),

\[
K_1 \max_{1 \leq j \leq n} |a_j| \leq \| \sum_{j=1}^n a_j x_j \| \leq K_2 \max_{1 \leq j \leq n} |a_j|.
\]

For each \( n \in \mathbb{N} \) define \( y_n \in H^0(T) \) by \( y_n(e^{i\theta}) = e^{in\theta} \). Then \( x_n \otimes y_n \in H^p(T, X) \), where \( (x_n \otimes y_n)(e^{i\theta}) = x_n e^{in\theta} \) and

\[
K_1 \max_{1 \leq j \leq n} |a_j| \leq \| \sum_{j=1}^n a_j (x_j \otimes y_j)(e^{i\theta}) \| \leq K_2 \max_{1 \leq j \leq n} |a_j|.
\]

Therefore

\[
K_1 \max_{1 \leq j \leq n} |a_j| \leq \| \sum_{j=1}^n a_j (x_j \otimes y_j) \|_p \leq K_2 \max_{1 \leq j \leq n} |a_j|.
\]

That is, \( (x_n \otimes y_n)_{n=1}^\infty \) is equivalent to the unit vector basis of \( c_0 \) in \( H^p(T, X) \). Now let \( (x_n^*)_{n=1}^\infty \) be a bounded sequence in \( X^* \) which is biorthogonal to \( (x_n)_{n=1}^\infty \).
and let \((y_n^*)^\infty_{n=1}\) be a sequence in \(L^\infty(T)\) defined by \(y_n^*(e^{i\theta}) = e^{-i\theta}\). Clearly, 
\((x_n ^* \otimes y_n ^*)^\infty_{n=1}\) is a sequence in \((H^p(T,X))^*\), and for each \(n \in \mathbb{N}\), 
\((x_n ^* \otimes y_n ^*)(x_n \otimes y_n) = 1\). Also, if \(f \in H^p(T,X)\), then \((x_n ^* \otimes y_n ^*)(f) = x_n ^*(\hat{f}(n))\) and \(x_n ^*(\hat{f}(n)) \to 0\) as \(n \to \infty\), since \(\|\hat{f}(n)\| \to 0\) as \(n \to \infty\). To see this, define \(S_f: L^\infty(T) \to X\) by

\[
S_f(g) = \int_0^{2\pi} g(e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi} \quad \text{for } g \in L^\infty(T).
\]

\(S_f\) is a compact linear operator [3], so \((\hat{f}(n))_{n=1}^\infty = \{S_f(e^{-i\theta})\}_{n=1}^\infty\) is a relatively compact subset of \(X\). If \(x^* \in X^*\), then

\[
x^*(\hat{f}(n)) = x^*S_f(e^{-i\theta}) = \int_0^{2\pi} x^*f(e^{i\theta})e^{-i\theta}\frac{d\theta}{2\pi} \to 0
\]
as \(n \to \infty\) since \(x^* f \in L^p(T)\) and the Riemann-Lebesgue lemma. Therefore \((\hat{f}(n))_{n=1}^\infty\) converges weakly to 0 and hence converges to 0 in norm.

Thus \((x_n ^* \otimes y_n ^*)^\infty_{n=1}\) is weak* null so \((x_n ^* \otimes y_n ^*)^\infty_{n=1}\) and \((x_n ^* \otimes y_n ^*)^\infty_{n=1}\) satisfy the conditions of the lemma, which completes the proof.

**Remark 1.** It is clear that this proof can be used in the following setting: Let \(G\) be a compact abelian group with normalized Haar measure on \(G\). Let \(\hat{G}\) be the dual group of \(G\), and let \(\Lambda\) be a subset of \(\hat{G}\). For \(1 \leq p \leq \infty\) and a complex Banach space \(X\), we define

\[
L^p_\Lambda(G,X) = \{f \in L^p(G,X): \hat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Lambda\}.
\]

If \(X\) contains a copy of \(c_0\), if \(1 \leq p < \infty\), and if \(\Lambda\) is infinite, then \(L^p_\Lambda(G,X)\) contains a complemented copy of \(c_0\). Note that if \(f \in L^1(G)\), then the net \((\hat{f}(\gamma))_{\gamma \in \Lambda}\) is an element of \(c_0(\Lambda)\) (see [6]).

**Remark 2.** The conclusion of the theorem does not hold true if \(H^p(T,X)\) is replaced by \(H^p(D,X)\). For example, consider \(H^p(D,\ell_\infty)\) for \(1 \leq p \leq \infty\). By a result of Dowling [4], \(H^p(D,\ell_\infty)\) is a dual Banach space for \(1 \leq p \leq \infty\). However, Bessaga and Pelczynski [1] have proved that \(c_0\) is never complemented in the dual of a Banach space. Therefore, \(H^p(D,\ell_\infty)\) does not contain complemented copies of \(c_0\). We know that \(H^p(T,\ell_\infty)\) is isomorphic to a subspace of \(H^p(D,\ell_\infty)\), so the results of this note show that \(H^p(T,\ell_\infty)\) is not isomorphic to a complemented subspace of \(H^p(D,\ell_\infty)\) when \(1 \leq p < \infty\).

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**References**


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