A NEW PROOF OF UNIQUENESS FOR
MULTIPLE TRIGONOMETRIC SERIES

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Abstract. Georg Cantor’s 1870 theorem that an everywhere convergent to zero trigonometric series has all its coefficients equal to zero is given a new proof. The new proof uses the first formal integral of the series, while Cantor’s proof used the second formal integral.

In 1870 Georg Cantor proved the following uniqueness theorem:

Theorem (Cantor [3]). If \( \{a_n\} \) and \( \{b_n\} \) are sequences of real numbers and if

\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]

converges at each \( x \) to 0, then the series vanishes identically; i.e., all its coefficients are 0.

Cantor’s proof used an idea of Riemann: That much of the behavior of (1) can be inferred from studying its formal second integral, \( \frac{1}{2}a_0 x^2 - \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)/n^2 \). For this proof see [7, p. 326], [1, pp. 1–4] or [3]. The idea of the present work is to use the first formal integral, \( L(x) := \frac{1}{2}a_0 x + \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)/n \). (\( L \) is for Lebesgue.) Zygmund points out that the difficulty in using \( L(x) \) is that \( L(x) \) need not converge everywhere even if the series (1) does. (For example \( \sum \sin nx/\log n \) converges everywhere but \( -\sum \cos n0/\log n \) diverges.) Nevertheless, here is a proof, dedicated to the would be extenders of Cantor’s theorem, which uses \( L \).

Proof. By the theorem of Cantor-Lebesgue ([7, p. 316], [1, Appendix 1] or [2]), \( a_n \) and \( b_n \) tend to 0 as \( n \) tends to \( \infty \) so that the coefficients of the series part of \( L \) are \( o(1/n) \), whence the sum of their squares is finite. By the Riesz-Fisher Theorem, this series represents an \( L^2 \) function. ([7, p. 127]) A theorem of Rajchman and Zygmund says that at every point the symmetric approximate derivative of \( L \) is equal to (the value of (1) which is) 0. ([7, p. 324]) We may also assume that \( L \) is approximately continuous at every
point of a $2\pi$-periodic set $E$ of full Lebesgue measure, since every measurable function is approximately continuous a.e. ([6, Vol. 2, p. 257]). Since $L(x)$ is approximately continuous and has non-negative symmetric derivative on $E$, by a recent elementary but ingenious and difficult result of C. Freiling and D. Rinne, $L$ is non-decreasing on $E$ ([4] and [5, Theorem 2]). Symmetrically $L$ is non-increasing on $E$, so that there is a constant $c$ with $L(x) = c$ for all $x$ in $E$. In other words, for all $x \in E$,

$$
-c + \sum_{n=1}^{\infty} \left( a_n \sin nx - b_n \cos nx \right)/n = -\frac{1}{2} a_0 x.
$$

The left side of equation (2) represents an $L^2$ function and is therefore Abel summable a.e. to that function ([7, p. 90, Equation 3.9 and p. 80, Equation 1.33]). At each point of Abel summability Tauber's original Tauberian theorem (recall the coefficients are $o(1/n)$) guarantees convergence ([7, p. 81]). Fix one such point $x_0$ which is also in $E$. Since equation (2) holds at $x_0$ and at $x_0 + 2\pi$ and the left side has the same value at both points, it follows that $a_0 = 0$. This means that the $L^2$ function represented by the left side of equation (2) is 0 a.e. Bessel's inequality ([7, p. 13, Equation 7.5]) gives that $-c + \sum (a_n^2 + b_n^2)/n^2 \leq 0$. The Theorem is proved.

Remark. The theorem of Freiling and Rinne which replaces the theorem of Schwarz and Cantor ([1, Appendix 2], [7, pp. 23 and 326], or [3]) that appears in the classical proof of the Theorem seems to avoid the maximum principle. However, Freiling and Rinne's present proof seems to require special properties of $\mathbb{R}^1$ that are not enjoyed by $\mathbb{R}^n$ for $n > 1$.

References

3. —, *Beweis, das eine für jeden reellen Wert von $x$ durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen lässt*, Crelles J. für Math. 72 (1870) 139–142; also in Gesammelte Abhandlungen, Georg Olms, Hildesheim, 1962, 80–83.

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