REPRESENTATIVES FOR FINITE SETS

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ABSTRACT. This paper considers a combinatorial problem by M. B. Nathanson [1], concerning simultaneous systems of representatives for two families of finite sets.

1. Introduction

Let \( \mathcal{S} = \{S_i\} \) be a family of nonempty sets. The set \( X \) is a system of representatives for \( \mathcal{S} \) if \( X \cap S_i \neq \emptyset \) for every \( S_i \) in \( \mathcal{S} \). If \( X \) is a system of representatives for \( \mathcal{S} \) but no proper subset of \( X \) is a system of representatives for \( \mathcal{S} \), then \( X \) is called a minimal system of representatives for \( \mathcal{S} \). By \( D(\mathcal{S}) \) we denote the number of minimal systems of representatives for \( \mathcal{S} \). Let \( |S| \) denote the cardinality of the set \( S \). If \( \mathcal{S} \) consists of \( s \) pairwise disjoint sets \( S_i \) with \( |S_i| = h \) for all \( i \), then \( D(\mathcal{S}) = h^s \).

Let \( \mathcal{S} = \{S_i\} \) and \( \mathcal{T} = \{T_j\} \) be two families of nonempty sets. A set \( X \) is called a simultaneous system of representatives for \( \mathcal{S} \) and \( \mathcal{T} \) if \( X \) is a minimal system of representatives for \( \mathcal{S} \) and \( X \) is also a system of representatives for \( \mathcal{T} \). \( N(\mathcal{S}, \mathcal{T}) \) denotes the number of the simultaneous systems of representatives for \( \mathcal{S} \) and \( \mathcal{T} \). The study of the numbers \( D(\mathcal{S}) \) and \( N(\mathcal{S}, \mathcal{T}) \) could be usefully applied to investigate minimal asymptotic bases in additive number theory [2].

In 1985, Nathanson [1] asked the following question:

Let \( h \geq 2 \) and \( k \geq 1 \). Does there exist a real number \( \mu = \mu(h,k) \in (0,1) \) such that

\[
N(\mathcal{S}, \mathcal{T}) \leq D(\mathcal{S}) \mu^l
\]

holds for any families \( \mathcal{S} \) and \( \mathcal{T} \) of sets satisfying the following properties?

(i) \( \mathcal{S} = \{S_i\} \) is a family of \( s \) nonempty, distinct sets \( S_i \) with \( |S_i| \leq h \) for all \( i \);

(ii) \( \mathcal{T} = \{T_j\} \) is a family of \( t \) nonempty, pairwise disjoint sets \( T_j \) with \( |T_j| \leq k \) for all \( j \);

(iii) \( S_i \) is not a subset of \( T_j \) for all \( i \) and \( j \).

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In this paper, it is proved that no such real number $\mu$ exists for any $h \geq 2$ and any $k \geq 1$. Adding some further restriction on $S$, we prove that such $\mu$ exists in a special case.

2. Main results

**Theorem 1.** Let $h \geq 2$ and $k \geq 1$. For any real number $\mu \in (0, 1)$, there exist two families of sets

$$\mathcal{S} = \{S_i; i = 1, \ldots, s\} \text{ and } \mathcal{T} = \{T_j; j = 1, \ldots, t\}$$

satisfying the following properties:

(i) $0 < |S_i| \leq h$ for all $i$;
(ii) $0 < |T_j| \leq k$ for all $j$;
(iii) $T_j \cap T_j' = \emptyset$ for all $j \neq j'$;
(iv) $S_i$ is not contained in $T_j$ for all $i$ and $j$;
(v) $N(\mathcal{S}, \mathcal{T}) > D(\mathcal{S})\mu^t$.

**Proof.** Let $\mu$ be any real number so that $0 < \mu < 1$. Let $t$ be an integer such that $\mu^t < 1/h$, and let $s = tk$. Let $a_1, \ldots, a_{h-1}, b_1, \ldots, b_s$ be $h - 1 + s$ different elements. Define

$$S_i = \{a_1, \ldots, a_{h-1}, b_i\}, \quad T_j = \{b_{(j-1)k+1}, \ldots, b_{jk}\}$$

for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, t$. Let

$$\mathcal{S} = \{S_i; i = 1, 2, \ldots, s\}, \quad \mathcal{T} = \{T_j; j = 1, 2, \ldots, t\}$$

It is clear that

$$N(\mathcal{S}, \mathcal{T}) = 1, \quad D(\mathcal{S}) = h - 1 + 1 = h.$$ 

Therefore we have

$$N(\mathcal{S}, \mathcal{T}) = 1 > h\mu^t = D(\mathcal{S})\mu^t,$$

which proves the theorem.

Theorem 1 means that the answer to the question is negative for any $h \geq 2$ and $k \geq 1$. However, we have the following result:

**Theorem 2.** Let $h \geq 2$. Suppose

(i) $\mathcal{S} = \{S_i\}$ is a family of nonempty, distinct sets $S_i$ with $|S_i| \leq h$ for all $i$;
(ii) Every $S_i$ intersects at most one $S_j$ in $\mathcal{S}$ other than $S_i$ itself;
(iii) $\mathcal{T} = \{T_j\}$ is a family of $t$ sets $T_j$ with $T_j = \{a_j\}$ for all $j$, where the $a_j$'s are distinct elements;
(iv) $S_i$ is not contained in $T_j$ for any $i$ and $j$.
Then

\[ N(\mathcal{S}, \mathcal{F}) \leq D(\mathcal{S})(1 - 1/h)^{(t-1)/2}. \]

Proof. By induction on \( t \) for any fixed \( s \). If \( t = 0 \), then

\[ N(\mathcal{S}, \mathcal{F}) = D(\mathcal{S}), \]

hence (1) holds for \( t = 0 \) and any \( s \). Let \( t \geq 1 \). Assume that (1) holds for any \( s \) and any \( t' < t \).

Let

\[ \mathcal{S} = \{ S_i : i = 1, 2, \ldots, s \} \quad \text{and} \quad \mathcal{F} = \{ T_j : j = 1, 2, \ldots, t \} \]

be two families of sets satisfying the conditions (i)-(iv). If there exists some \( T_j = \{ a_j \} \) such that \( a_j \in S_i \) for all \( i \), then \( N(\mathcal{S}, \mathcal{F}) = 0 \), hence (1) holds for \( t \) and any \( s \). Now we assume that

\[ S = \bigcup_{i=1}^{s} S_i \supseteq \{ a_1, \ldots, a_t \}. \]

We consider \( T_t = \{ a_t \} \). Then the following three cases may occur.

Case I. There exists an \( i' \) such that \( a_t \in S_{i'} \), where \( S_{i'} \cap S_i = \emptyset \) for all \( i \neq i' \). \( S_{i'} \not\subseteq T_t \) implies that \( |S_{i'}| \geq 2 \). It is readily verified that

\[ \mathcal{S}' = \mathcal{S} \setminus \{ S_{i'} \} \quad \text{and} \quad \mathcal{F}' = \{ T_j : j = 1, \ldots, t - 1 \} \]

satisfy the conditions (i)-(iv), and

\[ D(\mathcal{S}) = |S_{i'}|D(\mathcal{S}'). \]

If \( X \) is a simultaneous system of representatives for \( \mathcal{S} \) and \( \mathcal{F} \), then \( X' = X \setminus \{ a_t \} \) is a simultaneous system of representatives for \( \mathcal{S}' \) and \( \mathcal{F}' \). Conversely, if \( X' \) is a simultaneous system of representatives for \( \mathcal{S}' \) and \( \mathcal{F}' \), then \( X = X' \cup \{ a_t \} \) is a simultaneous system of representatives for \( \mathcal{S} \) and \( \mathcal{F} \). Therefore

\[ N(\mathcal{S}, \mathcal{F}) = N(\mathcal{S}', \mathcal{F}') \leq D(\mathcal{S}')(1 - 1/h)^{(t-1)/2} \]
\[ = (1/|S_{i'}|)D(\mathcal{S})(1 - 1/h)^{(t-1)/2} \]
\[ \leq \frac{1}{2}D(\mathcal{S})(1 - 1/h)^{(t-1)/2} \]
\[ < D(\mathcal{S})(1 - 1/h)^{t/2}. \]

Case II. There exists an \( i' \) such that \( a_t \in S_{i'} \cap S_{i''} \) for some \( i'' \). It follows from (ii) that

\[ (S_{i'} \cap S_{i''}) \cap S_i = \emptyset \]

for any \( i \neq i' \) and \( i \neq i'' \). Let

\[ |S_{i'} \cap S_{i''}| = r, \quad |S_{i'} \setminus S_{i''}| = u, \quad |S_{i''} \setminus S_{i'}| = v. \]
Since $a_i \in S_{i'}/S_{i''}$, it is clear that if $X$ is a simultaneous system of representatives for $\mathcal{S}'$ and $\mathcal{F}'$, then $X \setminus \{a_i\}$ is a simultaneous system of representatives for $\mathcal{S}' = \mathcal{S}' \setminus \{S_{i'}, S_{i''}\}$ and $\mathcal{F}' = \{T_j : j = 1, 2, \ldots, t - 1\}$.

Conversely, if $X'$ is a simultaneous system of representatives for $\mathcal{S}'$ and $\mathcal{F}'$, then $X = X' \cup \{a_i\}$ is a simultaneous system of representatives for $\mathcal{S}$ and $\mathcal{F}$. Hence

$$N(\mathcal{S}, \mathcal{F}) = N(\mathcal{S}', \mathcal{F}') .$$

It is clear that $D(\mathcal{S}) = (r + uv)D(\mathcal{S}')$. (iv) implies that $r + u \geq 2$ and $r + v \geq 2$, thus $1/(r + uv) \leq 1 - 1/h$. Therefore

$$N(\mathcal{S}, \mathcal{F}) = N(\mathcal{S}', \mathcal{F}') \leq D(\mathcal{S}')(1 - 1/h)^{(t-1)/2}$$

$$= \frac{1}{r + uv}D(\mathcal{S}')(1 - 1/h)^{(t-1)/2}$$

$$\leq (1 - 1/h)D(\mathcal{S}')(1 - 1/h)^{(t-1)/2}$$

$$< D(\mathcal{S}')(1 - 1/h)^{t/2} .$$

Case III. There exists an $i'$ such that

$$a_i \in S_{i'} \setminus S_{i''} \quad \text{and} \quad S_{i'} \cap S_{i''} \neq \emptyset$$

for some $i'' \neq i'$. Let

$$|S_{i'} \cap S_{i''}| = r, \quad |S_{i'} \setminus S_{i''}| = u, \quad |S_{i''} \setminus S_{i'}| = v .$$

It is clear that if there are two sets $T_j$ such that $T_j \subseteq S_{i'} \setminus S_{i''}$, then $N(\mathcal{S}, \mathcal{F}) = 0$, hence (1) holds. If there exists exactly one $T_j = \{a_j\}$ such that $a_j \in S_{i'} \setminus S_{i''}$, then any simultaneous system $X$ of representatives for $\mathcal{S}$ and $\mathcal{F}$ contains $a_i$ and $a_j$. Hence $X$ is a simultaneous system of representatives for $\mathcal{S}$ and $\mathcal{F}$ if and only if $X \setminus \{a_i, a_j\}$ is a simultaneous system of representatives for $\mathcal{S}' = \mathcal{S}' \setminus \{S_{i'}, S_{i''}\}$ and $\mathcal{F}' = \mathcal{F}' \setminus \{T_j, T_j\}$.

It is easily seen that

$$D(\mathcal{S}) = (r + uv)D(\mathcal{S}') .$$

Noticing that $r + uv \geq 2$, we have

$$N(\mathcal{S}, \mathcal{F}) = N(\mathcal{S}', \mathcal{F}') \leq D(\mathcal{S}')(1 - 1/h)^{(t-2)/2}$$

$$= \frac{1}{r + uv}D(\mathcal{S}')(1 - 1/h)^{(t-2)/2}$$

$$\leq \frac{1}{2}D(\mathcal{S}')(1 - 1/h)^{(t-2)/2}$$

$$= D(\mathcal{S}')(1 - 1/h)^{t/2} .$$

If there does not exist $T_j$ such that $a_j \in S_{i'} \setminus S_{i''}$, i.e., if $(S_{i'} \setminus S_{i''}) \cap T_j = \emptyset$ for all $j$, then any simultaneous system $X$ of representatives for $\mathcal{S}$ and $\mathcal{F}$
contains \( a_i \) and an element \( x \) of \( S_i \setminus S_{i'} \), hence \( X \setminus \{a_i, x\} \) is a simultaneous system of representatives for

\[
\mathcal{S}' = \{S_i \setminus S_j \setminus S_{i'} : j = 1, 2, \ldots, t - 1\}.
\]

Conversely, if \( X' \) is a simultaneous system of representatives for \( \mathcal{S}' \) and \( \mathcal{T}' \), then \( X = X' \cup \{a_i, x\} \) is a simultaneous system of representatives for \( \mathcal{S}' \) and \( \mathcal{T}' \) for any \( x \) in \( S_i \setminus S_{i'} \). It follows the fact that \( r \geq 1, 0 \leq u \leq h - 1 \) and \( v \geq 1 \) that

\[
\frac{v}{r + uv} \leq 1 - \frac{1}{h}.
\]

Therefore

\[
N(\mathcal{S}, \mathcal{T}) = |S_i \setminus S_{i'}|N(\mathcal{S}', \mathcal{T}') \\
\leq V D(\mathcal{S}') (1 - 1/h)^{(t-1)/2} \\
= \frac{v}{r + uv} D(\mathcal{S}) (1 - 1/h)^{(t-1)/2} \\
\leq (1 - 1/h) D(\mathcal{S}) (1 - 1/h)^{(t-1)/2} \\
< D(\mathcal{S}) (1 - 1/h)^{t/2}.
\]

This completes the proof.

References


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