ON SUPERPOSITION OF FUNCTIONS OF BOUNDED $\varphi$-VARIATION

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Abstract. J. Ciernoczolowski and W. Orlicz in [1] have obtained some results concerning superpositions of functions of bounded $\varphi$-variation. In this note we show that the assumption in Theorem 1 of [1] that $\psi$ satisfies $\Delta_2$ condition may be dropped. Moreover, Theorem 2.B of [1] is extended to a stronger version.

A function $\varphi : (0, \infty) \rightarrow (0, \infty)$ is called a $\varphi$-function if it is continuous, nondecreasing and such that $\varphi(0) = 0, \varphi(u) > 0$ for $u > 0$ and $\varphi(u) \rightarrow \infty$ for $u \rightarrow \infty$. A $\varphi$-function $\varphi$ satisfies condition $\Delta_2$ for small $u$ if $\limsup \varphi(2u)/\varphi(u) < \infty$ for $u \rightarrow 0^+$. We denote by $X$ the vector space of real functions defined on a closed interval $(a, b)$ which vanish at $a$. For $x \in X$, we denote by $\text{osc}(x ; (a, b))$ the oscillation of $x$ on $(a, b)$.

Let $\varphi$ be a $\varphi$-function and $A$ be a subset of real numbers. A finite subset $\pi$ of $A$ with the natural order we will call a partition of $A$. In general, we will write a non-empty partition $\pi$ of $A$ in form of an increasing finite sequence $(t_i)_{i=1}^n$. For a real function $x$ defined on $A$ and for a partition $\pi$ of $A$ we define

$$\var_{\varphi}(x ; \pi) = \begin{cases} 0 & \text{if } \text{card } \pi \leq 1 \\ \sum_{i=1}^{n-1} \varphi(|x(t_{i+1}) - x(t_i)|) & \text{if } \text{card } \pi \geq 2 \end{cases}$$

The value $\var_{\varphi}(x ; A) = \sup_{\pi} \var_{\varphi}(x ; \pi)$, where the supremum is taken over all partitions of $A$, is called a $\varphi$-variation of $x$ on $A$. If the $\varphi$-variation of $x$ is finite then we say $x$ is of bounded $\varphi$-variation. It is easy to see that if $A$ is a closed interval $(a, b)$ then the above definition of $\varphi$-variation of $x$ on $A$ is equivalent to the classical one ([4], p. 582), which was used in [1]. The class of all functions $x \in X$ of bounded $\varphi$-variation is denoted by $V_{\varphi}(a, b)$.

J. Ciernoczolowski and W. Orlicz have stated in [1] the following theorem concerning superpositions of functions of bounded $\varphi$-variation.

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Theorem ([1], Theorem 1). Let $\varphi$ be an arbitrary $\varphi$-function, $\psi$ a $\varphi$-function satisfying $\Delta_2$ for small $u$. Let $F_n$ be real functions on $(-\infty, \infty)$, $F_n(0) = 0$, $n = 1, 2, \ldots$. Then the following are equivalent:

(a) $\sup_n \varphi(F_n(x); (a, b)) < \infty$ for $x \in V_{\varphi}(a, b)$;
(b) for every $r > 0$ there exists a constant $C_r > 0$ such that the inequality $\psi(|F_n(u_1) - F_n(u_2)|) \leq C_r \varphi(|u_1 - u_2|)$ holds for $u_1, u_2 \in (-r, r)$, $n = 1, 2, \ldots$.

We will show that the assumption $\psi$ satisfies condition $\Delta_2$ for small $u$ may be dropped. Namely, one has

Theorem 1. Let $(F_n)$ be a sequence of real functions defined on $(-\infty, \infty)$ and $F_n(0) = 0$ for $n = 1, 2, \ldots$. For every pair $\varphi, \psi$ of $\varphi$-functions the following statements are equivalent:

(i) For every sequence $(x_n)$ of functions of $X$ if $\sup_n \varphi(x_n; (a, b)) < \infty$ then $\sup_n \varphi(F_n(x_n); (a, b)) < \infty$.
(ii) If $x \in V_{\varphi}(a, b)$ then $\sup_n \varphi(F_n(x); (a, b)) < \infty$.
(iii) For every $r > 0$ there exists a constant $C_r > 0$ such that the inequality $\psi(|F_n(u_1) - F_n(u_2)|) \leq C_r \varphi(|u_1 - u_2|)$ holds for $u_1, u_2 \in (-r, r)$, $n = 1, 2, \ldots$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

(iii) $\Rightarrow$ (i). If $\sup_n \varphi(x_n; (a, b))$ is finite then $r = \sup_n \text{osc}(x_n; (a, b))$ is finite and by (iii)

$$\sup_n \varphi(F_n(x_n); (a, b)) \leq C_r \sup_n \varphi(x_n; (a, b)) < \infty.$$ 

(ii) $\Rightarrow$ (iii). Assume (ii) holds. Then for every $r = 1, 2, \ldots$ the functions $F_n$ are uniformly bounded in common in $(-r, r)$ (see [1], proof of Theorem 1). If (iii) does not hold then there exist an integer $r > 0$, a nondecreasing sequence $(n_i)$ of indices, subintervals $(u_i, v_i)$ of $(-r, r)$ such that

$$d_i = \frac{\psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|)}{\varphi(v_i - u_i)} \to \infty.$$ 

Since $F_{n_i}$ are uniformly bounded in common in $(-r, r)$, we have $u_i - v_i \to 0$. Without loss of generality we may assume that sequences $(u_i)$ and $(v_i)$ are convergent to a point $w$ of $(-r, r)$. Passing, if necessary, to a partial sequence, we state that one of the following cases holds:

(A) $u_i \geq w$ for all $i$;

(B) $v_i \leq w$ for all $i$;

(C) $u_i \leq v_{i+1} < u < v_{i+1} \leq v_i$ for all $i$.

If (A) holds then as in ([1], proof of Theorem 1) we construct a function $x \in V_{\varphi}(a, b)$ such that $\sup_n \varphi(F_n(x); (a, b)) = \infty$. If (B) holds then, setting $G_n(u) = F_n(-u)$ for $u \in (-\infty, \infty)$ and $n = 1, 2, \ldots$, the case (A) holds for $G_n$. Therefore, there exists a function $x \in V_{\varphi}(a, b)$ such that $\sup_n \varphi(F_n(x); (a, b)) = \sup_n \varphi(G_n(x); (a, b)) = \infty$.
If (C) holds then there exists an increasing sequence \( i_j \) of positive integers such that \( \varphi(v_i - u_i) \leq 2^{-j-2} \) and \( d_i \geq 2 \cdot 4^j \) for \( j = 1, 2, \cdots \). We set
\[
s_j = \min\{s \in \{1, 2, \ldots\}: (2s - 1)\varphi(v_i - u_i) \geq 2^{-j-1}\},
\]
\[
m_0 = 0, \quad m_j = \sum_{k=1}^{j} 2s_k \quad \text{for } j = 1, 2, \ldots.
\]
Given a decreasing sequence \( (t_k) \) of points of \((a, b)\) convergent to \( a \) with \( t_1 = b \) and setting \( x(a) = 0 \) and for \( j = 1, 2, \ldots; \ k = m_j + 1, \ldots, m_j; \ t \in (t_{k+1}, t_k) \)
\[
x(t) = \begin{cases} v, & \text{for odd } k \\ u, & \text{for even } k \end{cases},
\]
we obtain a regulated function \( x \in X \).

Now, we shall prove that \( x \in V_{\varphi}(a, b) \). By ([3], Lemma 1.1) it is enough to show that \( \varphi(x; (a, b)) < \infty \). Observe that for every \( \varphi \)-function \( x \) and every function \( F: (-\infty, \infty) \to (-\infty, 0) \) we have
\[
(a) \varphi(F(x); (t_{1+m_j}, t_{1+m_j-1})) = (2s_j - 1)\varphi(F(v_i) - F(u_i)) \quad \text{for } j = 1, 2, \ldots,
\]
\[
(b) \varphi(x; (t_{1+m_k}, b)) = \sum_{j=1}^{k} \varphi(x; (t_{1+m_j}, t_{1+m_j-1}))
\]
\[
+ \sum_{j=1}^{k-1} \varphi(v_i - u_i) \quad \text{for } k = 1, 2, \ldots.
\]
Thus, for a partition \( \pi = (r_i)_{i=1}^{n} \) of \((a, b)\) if \( t_{1+m_k} < r_1 \) then
\[
\varphi(x; \pi) \leq \varphi(x; (t_{1+m_k}, b)) \leq \sum_{j=1}^{n} \varphi(v_i - u_i) \leq \frac{5}{4}.
\]
Therefore \( x \in V_{\varphi}(a, b) \).

For \( j = 1, 2, \ldots \), by (a) we have
\[
\varphi(F_n(x); (a, b)) \geq \varphi(F_{n_j}(x); (t_{1+m_j}, t_{1+m_j-1})) \geq (2s_j - 1)\varphi(F_{n_j}(v_i) - F_{n_j}(u_i)) \geq (2s_j - 1) \cdot 2 \cdot 4^j \varphi(v_i - u_i) \geq 2^j.
\]
Thus \( \sup_n \varphi(F_n(x); (a, b)) = \infty \) and we get a contradiction. \( \Box \)
If we omit the assumption that \( \psi \) satisfies the \( \Delta_2 \) condition then all consequences of ([1], Theorem 1) remain true, except Theorem 2.B.

For a \( \varphi \)-function \( \varphi \) and a real function \( F \) defined on \((-\infty, \infty)\) we will write \( F \in \text{GL}_{\varphi} \) if \( F(0) = 0 \) and \( F \) satisfies the following generalized Lipschitz condition: for every \( k > 0 \) there exists a constant \( C_k > 0 \) such that \( \varphi(|F(u) - F(v)|) \leq C_k \varphi(|u - v|) \) for \( u, v \in (-k, k) \). It is easy to see that functions from \( \text{GL}_{\varphi} \) are continuous. If \( \varphi(u) = u \) then we will write \( \text{GL} \) instead \( \text{GL}_{\varphi} \). Using this notation the Theorem 2.A of [1] may be written as follows: \( F(V_{\varphi}'(a, b)) \subset V_{\varphi}(a, b) \) iff \( F \in \text{GL}_{\varphi} \).

J. Ciemnoczolowski and W. Orlicz have formulated in [1] a sufficient condition for the equality \( \text{GL}_{\varphi} = \text{GL} \). Namely, they have proved the following

Theorem ([1], Theorem 2.B). Let \( \varphi \) be a strictly increasing \( \varphi \)-function such that \( \varphi \) and \( \varphi^{-1} \) satisfy condition \( \Delta_2 \) for small \( u \). Then \( \text{GL}_{\varphi} = \text{GL} \).

Theorems 2 and 3 jointly allow to formulate the necessary and sufficient condition for the equality \( \text{GL}_{\varphi} = \text{GL} \).

**Theorem 2.** The inclusion \( \text{GL}_{\varphi} \subset \text{GL} \) holds if and only if \( \varphi \) satisfies the condition

\[
\text{(E) for every } c > 0 \text{ there exists a number } r > 0 \text{ such that } \\
\limsup_{u \to 0^+} \frac{\varphi(ru)}{\varphi(u)} > c.
\]

To prove this theorem we need a simple lemma.

**Lemma.** Let \( F \) be a real function defined on \((a, b)\). Then for every positive integer \( n \) there exist points \( s, t \) of \((a, b)\) such that

\[
t - s = \frac{b - a}{n}
\]

and

\[
\frac{|F(t) - F(s)|}{t - s} \geq \frac{|F(b) - F(a)|}{b - a}.
\]

**Proof.** If for some integer \( n > 0 \) and every pair \( s, t \) of points of \((a, b)\), satisfying (1), the inequality

\[
\frac{|F(t) - F(s)|}{t - s} < \frac{|F(b) - F(a)|}{b - a}
\]

holds, then setting \( s_k = a + (k - 1)(b - a)/n \) and \( t_k = a + k(b - a)/n \) for \( k = 1, \ldots, n \), we have

\[
|F(b) - F(a)| \leq \sum_{k=1}^{n} |F(t_k) - F(s_k)|
\]

\[
< \frac{|F(b) - F(a)|}{b - a} \sum_{k=1}^{n} (t_k - s_k) = |F(b) - F(a)|
\]

and we get a contradiction. \( \Box \)
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Proof of Theorem 2. First, suppose that $\varphi$ satisfies the condition (E) and that $F \in \text{GL}_\varphi \setminus \text{GL}$. Let $m$ be a positive number such that for every $k > 0$ there exists a subinterval $(u_k, v_k)$ of $(-m, m)$ such that $|F(v_k) - F(u_k)| \geq k(v_k - u_k)$. Let $C > 0$ be such that $\varphi(|F(u) - F(v)|) \leq C\varphi(|u - v|)$ for $u, v \in (-m, m)$.

For some $r > 0$ we have
\[
\limsup_{u \to 0^+} \frac{\varphi(ru)}{\varphi(u)} > C + 1
\]
and there exists a subinterval $(u, v)$ of $(-m, m)$ such that $|F(v) - F(u)| \geq (r + 1)(v - u)$. Since $F$ is continuous, it follows that there exists an $\varepsilon \in (0, v - u)$ such that
\[
|F(v) - F(s)| \geq r(v - s) \quad \text{for } s \in (u, u + \varepsilon).
\]
Choosing a number $w \in (0, \varepsilon)$ so that
\[
\varphi(rw) > (C + 1)\varphi(w),
\]
we have $v - lw \in (u, u + \varepsilon)$ for some integer $l > 0$. Thus, by $(+) |F(v) - F(v - lw)| \geq rlw$ and therefore, by our Lemma there exists a subinterval $(u', v')$ of $(v - lw, v)$ such that $v' - u' = w$ and $|F(v') - F(u')| \geq r(v' - u')$. Thus,
\[
\varphi(rw) = \varphi(r(v' - u')) \leq \varphi(|F(v') - F(u')|) \leq C\varphi(v' - u') < (C + 1)\varphi(w),
\]
which contradicts $(++)$. So if $\varphi$ satisfies the condition (E) then $\text{GL}_\varphi \subset \text{GL}$.

Conversely, suppose that $\varphi$ does not satisfy (E). Then there exists a constant $c > 1$ such that for every $n = 1, 2, \ldots$ there exists a number $v_n > 0$ such that $\varphi((n + 1)u) \leq c\varphi(u)$ for $u \in (0, v_n)$. For a sequence $(u_n)$ of positive numbers such that $u_1 \leq 1$, $u_n < v_n$ and $(n + 1)u_{n+1} < nu_n$ for $n = 1, 2, \ldots$, the series $\sum_n (-1)^n nu_n$ is convergent. Now, we define a real function on $(-\infty, \infty)$, setting
\[
t_n = -\sum_{i=n}^{\infty} u_i \quad \text{for } n = 1, 2, \ldots;
\]
\[
F(t) = 0 \quad \text{for } t \geq 0; \quad F(t) = \sum_n (-1)^n nu_n \quad \text{for } t < t_1;
\]
\[
F(t_n) = \sum_{i=n}^{\infty} (-1)^i u_i \quad \text{for } n = 1, 2, \ldots.
\]
Finally, we define $F$ to be a linear function on each $(t_n, t_{n+1})$. Observe that for every integer $n > 0$ and every $u \in (u_{n+1}, u_n)$ we have $|F(d + u) - F(d)| \leq (n + 1)u_{n+1}$ for $d \geq t_{n+1}$ and $|F(d + u) - F(d)| \leq nu$ for $d < t_{n+1}$. Hence for every real number $d$
\[
|F(d + u) - F(d)| \leq (n + 1)u \quad \text{for } u \in (u_{n+1}, u_n).
\]

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More, observe that for \( u \geq u_1 \) and for every real number \( d \)

\[
|F(d + u) - F(d)| \leq u_1.
\]

Given two different real numbers \( w, w' \), if \( |w - w'| \geq u_1 \) then by

\[
\varphi(F(w) - F(w')) \leq \varphi(u_1) \leq c\varphi(|w - w'|).
\]

If \( |w - w'| \in (u_{n+1}, u_n) \) then by \( \varphi \)

\[
\varphi(|F(w) - F(w')|) \leq \varphi((n+1)|w - w'|) \leq c\varphi(|w - w'|),
\]

because \( u_n < v_n \). We have proved that \( F \in \text{GL}_\varphi \). Finally, for \( n = 1, 2, \ldots \)

we have \( t_n \in (-2, 2) \) and

\[
\frac{|F(t_{n+1}) - F(t_n)|}{t_{n+1} - t_n} = \frac{nu_n}{u_n} = n.
\]

Thus \( F \not\in \text{GL} \). So if \( \text{GL}_\varphi \subset \text{GL} \) then \( \varphi \) satisfies the condition (E). \( \square \)

**Theorem 3.** The inclusion \( \text{GL} \subset \text{GL}_\varphi \) holds if and only if \( \varphi \) satisfies the condition \( \Delta_2 \) for small \( u \).

**Proof.** First, assume that \( \varphi \) satisfies \( \Delta_2 \) for small \( u \) and \( F \in \text{GL} \). Given \( m > 0 \), there exists a constant \( C > 0 \) such that \( \varphi(|F(u) - F(v)|) \leq \varphi(C|u - v|) \)

for \( u, v \in (-m, m) \). Because \( \varphi \) satisfies \( \Delta_2 \) for small \( u \), there exists a constant \( K_m > 0 \) such that \( \varphi(Cw) \leq K_m \varphi(w) \) for \( 0 \leq w \leq 2m \) (cf. [2], 1.02). Thus,

\[
\varphi(|F(u) - F(v)|) \leq K_m \varphi(|u - v|) \quad \text{for} \quad u, v \in (-m, m).
\]

It follows that \( F \in \text{GL}_\varphi \).

Conversely, suppose that \( \varphi \) does not satisfy the condition \( \Delta_2 \) for small \( u \).

Then

\[
\limsup_{u \to 0^+} \frac{\varphi(2u)}{\varphi(u)} = \infty.
\]

and it is easy to see that for \( F(u) = 2u \) we have \( F \in \text{GL} \) and \( F \not\in \text{GL}_\varphi \). \( \square \)

The following result is just exactly a generalization of Theorem 2.B of [1].

**Corollary.** The identity \( \text{GL} = \text{GL}_\varphi \) holds if and only if \( \varphi \) satisfies the conditions (E) and \( \Delta_2 \) for small \( u \).

**References**


