ON SUPERPOSITION OF FUNCTIONS OF BOUNDED $\varphi$-VARIATION

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Abstract. J. Ciemnoczolowski and W. Orlicz in [1] have obtained some results concerning superpositions of functions of bounded $\varphi$-variation. In this note we show that the assumption in Theorem 1 of [1] that $\psi$ satisfies $\Delta_2$ condition may be dropped. Moreover, Theorem 2.B of [1] is extended to a stronger version.

A function $\varphi : (0, \infty) \to (0, \infty)$ is called a $\varphi$-function if it is continuous, nondecreasing and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ for $u \to \infty$. A $\varphi$-function $\varphi$ satisfies condition $\Delta_2$ for small $u$ if $\limsup \varphi(2u)/\varphi(u) < \infty$ for $u \to 0^+$. We denote by $X$ the vector space of real functions defined on a closed interval $\langle a, b \rangle$ which vanish at $a$. For $x \in X$, we denote by $\text{osc}(x; \langle a, b \rangle)$ the oscillation of $x$ on $\langle a, b \rangle$.

Let $\varphi$ be a $\varphi$-function and $A$ be a subset of real numbers. A finite subset $\pi$ of $A$ with the natural order we will call a partition of $A$. In general, we will write a non-empty partition $\pi$ of $A$ in form of an increasing finite sequence $(t_i)_{i=1}^n$. For a real function $x$ defined on $A$ and for a partition $\pi$ of $A$ we define

$$\text{var}_\varphi(x; \pi) = \begin{cases} 
0 & \text{if } \text{card } \pi \leq 1 \\
\sum_{i=1}^{n-1} \varphi(|x(t_{i+1}) - x(t_i)|) & \text{if } \text{card } \pi \geq 2
\end{cases}$$

The value $\text{var}_\varphi(x; A) = \sup_\pi \text{var}_\varphi(x; \pi)$, where the supremum is taken over all partitions of $A$, is called a $\varphi$-variation of $x$ on $A$. If the $\varphi$-variation of $x$ is finite then we say $x$ is of bounded $\varphi$-variation. It is easy to see that if $A$ is a closed interval $\langle a, b \rangle$ then the above definition of $\varphi$-variation of $x$ on $A$ is equivalent to the classical one ([4], p. 582), which was used in [1]. The class of all functions $x \in X$ of bounded $\varphi$-variation is denoted by $V_\varphi(\langle a, b \rangle)$.

J. Ciemnoczolowski and W. Orlicz have stated in [1] the following theorem concerning superpositions of functions of bounded $\varphi$-variation.

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Theorem ([1], Theorem 1). Let \( \varphi \) be an arbitrary \( \varphi \)-function, \( \psi \) a \( \varphi \)-function satisfying \( \Delta_2 \) for small \( u \). Let \( F_n \) be real functions on \((-\infty, \infty)\), \( F_n(0) = 0 \), \( n = 1, 2, \ldots \). Then the following are equivalent:

(a) \( \sup_n \varphi(F_n(x);(a,b)) < \infty \) for \( x \in V_{\varphi}(a,b) \);

(b) for every \( r > 0 \) there exists a constant \( C_r > 0 \) such that the inequality
\( \psi(|F_n(u_1) - F_n(u_2)|) \leq C_r \varphi(|u_1 - u_2|) \) holds for \( u_1, u_2 \in (-r,r), \ n = 1, 2, \ldots \).

We will show that the assumption \( \psi \) satisfies condition \( \Delta_2 \) for small \( u \) may be dropped. Namely, one has

**Theorem 1.** Let \( (F_n) \) be a sequence of real functions defined on \((-\infty, \infty)\) and \( F_n(0) = 0 \) for \( n = 1, 2, \ldots \). For every pair \( \varphi, \psi \) of \( \varphi \)-functions the following statements are equivalent:

(i) For every sequence \( (x_n) \) of functions of \( X \) if \( \sup_n \varphi(x_n; (a,b)) < \infty \) then \( \sup_n \varphi(F_n(x_n); (a,b)) < \infty \).

(ii) If \( x \in V_{\varphi}(a,b) \) then \( \sup_n \varphi(F_n(x); (a,b)) < \infty \).

(iii) For every \( r > 0 \) there exists a constant \( C_r > 0 \) such that the inequality
\( \psi(|F_n(u_1) - F_n(u_2)|) \leq C_r \varphi(|u_1 - u_2|) \) holds for \( u_1, u_2 \in (-r,r), \ n = 1, 2, \ldots \).

**Proof.** The implication (i) \( \Rightarrow \) (ii) is obvious.

(iii) \( \Rightarrow \) (i). If \( \sup_n \varphi(x_n; (a,b)) \) is finite then \( r = \sup_n \text{osc}(x_n; (a,b)) \) is finite and by (iii)
\[ \sup_n \varphi(F_n(x_n); (a,b)) \leq C_r \sup_n \varphi(x_n; (a,b)) < \infty. \]

(ii) \( \Rightarrow \) (iii). Assume (ii) holds. Then for every \( r = 1, 2, \ldots \) the functions \( F_n \) are uniformly bounded in common in \((-r,r)\) (see [1], proof of Theorem 1). If (iii) does not hold then there exist an integer \( r > 0 \), a nondecreasing sequence \( (n_i) \) of indices, subintervals \( (u_i, v_i) \) of \((-r,r)\) such that
\[ d_i = \frac{\psi(|F_{n_i}(v_i) - F_{n_i}(u_i)|)}{\varphi(v_i - u_i)} \to \infty. \]

Since \( F_{n_i} \) are uniformly bounded in common in \((-r,r)\), we have \( v_i - u_i \to 0 \). Without loss of generality we may assume that sequences \( (u_i) \) and \( (v_i) \) are convergent to a point \( w \) of \((-r,r)\). Passing, if necessary, to a partial sequence, we state that one of the following cases holds:

(A) \( u_i \geq w \) for all \( i \);

(B) \( v_i \leq w \) for all \( i \);

(C) \( u_i \leq u_{i+1} < w < v_{i+1} \leq v_i \) for all \( i \).

If (A) holds then as in ([1], proof of Theorem 1) we construct a function \( x \in V_{\varphi}(a,b) \) such that \( \sup_n \varphi(F_n(x); (a,b)) = \infty \). If (B) holds then, setting \( G_n(u) = F_n(-u) \) for \( u \in (-\infty, \infty) \) and \( n = 1, 2, \ldots \), the case (A) holds for \( G_n \). Therefore, there exists a function \( x \in V_{\varphi}(a,b) \) such that\[ \sup_n \varphi(F_n(x); (a,b)) = \sup_n \varphi(G_n(x); (a,b)) = \infty. \]
If (C) holds then there exists an increasing sequence \( i_j \) of positive integers such that \( \varphi(v_{i_j} - u_{i_j}) \leq 2^{-j-2} \) and \( d_{i_j} \geq 2 \cdot 4^j \) for \( j = 1, 2, \ldots \). We set

\[
s_j = \min\{s \in \{1, 2, \ldots\} : (2s - 1)\varphi(v_{i_j} - u_{i_j}) \geq 2^{-j-1}\},
\]

\[
m_0 = 0, \quad m_j = \sum_{k=1}^{j} 2s_k \quad \text{for} \quad j = 1, 2, \ldots.
\]

Given a decreasing sequence \( (t_k) \) of points of \( (a, b) \) convergent to \( a \) with \( t_1 = b \) and setting \( x(a) = 0 \) and for \( j = 1, 2, \ldots; k = m_{j-1} + 1, \ldots, m_j; t \in (t_{k+1}, t_k) \)

\[x(t) = \begin{cases} v_{i_j} & \text{for odd } k \\ u_{i_j} & \text{for even } k \end{cases},\]

we obtain a regulated function \( x \in X \).

Now, we shall prove that \( x \in V_\varphi(a, b) \). By ([3], Lemma 1.1) it is enough to show that \( \varphi_x(x; (a, b)) < \infty \). Observe that for every \( \varphi \)-function \( x \) and every function \( F: (-\infty, \infty) \rightarrow (-\infty, \infty) \) we have

(a) \( \varphi_x(F(x); (t_{1+m_j}, t_{1+m_{j-1}})) = (2s_j - 1)\chi(|F(v_{i_j}) - F(u_{i_j})|) \)

for \( j = 1, 2, \ldots, \)

(b) \( \varphi_x(x; (t_{1+m_k}, b)) = \sum_{j=1}^{k} \varphi_x(x; (t_{1+m_j}, t_{1+m_{j-1}})) \\
+ \sum_{j=1}^{k-1} \varphi(v_{i_{j+1}} - u_{i_j}) \quad \text{for} \quad k = 1, 2, \ldots. \)

Thus, for a partition \( \pi = (r_i)_{i=1}^n \) of \( (a, b) \) if \( t_{1+m_k} < r_1 \) then

\[
\varphi_x(x; \pi) \leq \varphi_x(x; (t_{1+m_k}, b)) \\
\leq \sum_j (2s_j - 1)\varphi(v_{i_j} - u_{i_j}) + \sum_j \varphi(v_{i_{j+1}} - u_{i_j}) \\
\leq \sum_j 2^{-j} + \sum_j \varphi(v_{i_j} - u_{i_j}) \leq \frac{5}{4}.
\]

Therefore \( x \in V_\varphi(a, b) \).

For \( j = 1, 2, \ldots, \) by (a) we have

\[
\varphi_x(F_{n_{i_j}}(x); (a, b)) \geq \varphi_x(F_{n_{i_j}}(x); (t_{1+m_j}, t_{1+m_{j-1}})) \\
= (2s_j - 1)\varphi(|F_{n_{i_j}}(v_{i_j}) - F_{n_{i_j}}(u_{i_j})|) \\
\geq (2s_j - 1) \cdot 2 \cdot 4^j \varphi(v_{i_j} - u_{i_j}) \geq 2^j.
\]

Thus \( \sup_n \varphi_x(F_n(x); (a, b)) = \infty \) and we get a contradiction. \( \square \)
If we omit the assumption that $\psi$ satisfies the $\Delta_2$ condition then all consequences of ([1], Theorem 1) remain true, except Theorem 2.B.

For a $\varphi$-function $\varphi$ and a real function $F$ defined on $(-\infty, \infty)$ we will write $F \in \text{GL}_\varphi$ if $F(0) = 0$ and $F$ satisfies the following generalized Lipschitz condition: for every $k > 0$ there exists a constant $C_k > 0$ such that $\varphi(|F(u) - F(v)|) \leq C_k \varphi(|u - v|)$ for $u, v \in (-k, k)$. It is easy to see that functions from $\text{GL}_\varphi$ are continuous. If $\varphi(u) = u$ then we will write $\text{GL}$ instead $\text{GL}_\varphi$. Using this notation the Theorem 2.A of [1] may be written as follows: $F(V_\varphi(a, b)) \subset V_\varphi(a, b)$ iff $F \in \text{GL}_\varphi$.

J. Ciemnoczolowski and W. Orlicz have formulated in [1] a sufficient condition for the equality $\text{GL}_\varphi = \text{GL}$. Namely, they have proved the following

**Theorem** ([1], Theorem 2.B). Let $\varphi$ be a strictly increasing $\varphi$-function such that $\varphi$ and $\varphi^{-1}$ satisfy condition $\Delta_2$ for small $u$. Then $\text{GL}_\varphi = \text{GL}$.

Theorems 2 and 3 jointly allow to formulate the necessary and sufficient condition for the equality $\text{GL}_\varphi = \text{GL}$.

**Theorem 2.** The inclusion $\text{GL}_\varphi \subset \text{GL}$ holds if and only if $\varphi$ satisfies the condition

\[(E)\; \text{for every } c > 0 \text{ there exists a number } r > 0 \text{ such that} \limsup_{u \to 0^+} \frac{\varphi(ru)}{\varphi(u)} > c.\]

To prove this theorem we need a simple lemma.

**Lemma.** Let $F$ be a real function defined on $(a, b)$. Then for every positive integer $n$ there exist points $s, t$ of $(a, b)$ such that

(1) $t - s = \frac{b - a}{n}$

and

(2) $\frac{|F(t) - F(s)|}{t - s} \geq \frac{|F(b) - F(a)|}{b - a}$.

**Proof.** If for some integer $n > 0$ and every pair $s, t$ of points of $(a, b)$, satisfying (1), the inequality

$$\frac{|F(t) - F(s)|}{t - s} < \frac{|F(b) - F(a)|}{b - a}$$

holds, then setting $s_k = a + (k - 1)(b - a)/n$ and $t_k = a + k(b - a)/n$ for $k = 1, \ldots, n$, we have

$$|F(b) - F(a)| \leq \sum_{k=1}^n |F(t_k) - F(s_k)|$$

$$< \frac{|F(b) - F(a)|}{b - a} \sum_{k=1}^n (t_k - s_k) = |F(b) - F(a)|$$

and we get a contradiction. $\square$
Proof of Theorem 2. First, suppose that $\phi$ satisfies the condition (E) and that $F \in \text{GL}_\phi \setminus \text{GL}$. Let $m$ be a positive number such that for every $k > 0$ there exists a subinterval $(u_k, v_k)$ of $(-m, m)$ such that $|F(v_k) - F(u_k)| \geq k(v_k - u_k)$. Let $C > 0$ be such that $\phi(|F(u) - F(v)|) \leq C\phi(|u - v|)$ for $u, v \in (-m, m)$.

For some $r > 0$ we have

$$\limsup_{u \to 0^+} \frac{\phi(ru)}{\phi(u)} > C + 1$$

and there exists a subinterval $(u, v)$ of $(-m, m)$ such that $|F(v) - F(u)| \geq (r + 1)(v - u)$. Since $F$ is continuous, it follows that there exists an $\epsilon \in (0, v - u)$ such that

$$(+) \quad |F(v) - F(s)| \geq r(v - s) \quad \text{for} \quad s \in (u, u + \epsilon).$$

Choosing a number $w \in (0, \epsilon)$ so that

$$(++) \quad \phi(rw) > (C + 1)\phi(w),$$

we have $v - lw \in (u, u + \epsilon)$ for some integer $l > 0$. Thus, by $(+)$ $|F(v) - F(v - lw)| \geq rlw$ and therefore, by our Lemma there exists a subinterval $(v', v')$ of $(v - lw, v)$ such that $v' - u' = w$ and $|F(v') - F(u')| \geq r(v' - u')$. Thus,

$$\phi(rw) = \phi(r(v' - u')) \leq \phi(|F(v') - F(u')|) \leq C\phi(v' - u') < (C + 1)\phi(w),$$

which contradicts $(++)$. So if $\phi$ satisfies the condition (E) then $\text{GL}_\phi \subset \text{GL}$.

Conversely, suppose that $\phi$ does not satisfy (E). Then there exists a constant $c > 1$ such that for every $n = 1, 2, \ldots$ there exists a number $v_n > 0$ such that $\phi((n + 1)u) \leq c\phi(u)$ for $u \in (0, v_n)$. For a sequence $(u_n)$ of positive numbers such that $u_1 \leq 1$, $u_n < v_n$ and $2(n + 1)u_{n+1} < nu_n$ for $n = 1, 2, \ldots$, the series $\sum_n (-1)^n nu_n$ is convergent. Now, we define a real function on $(-\infty, \infty)$, setting

$$t_n = -\sum_{i=n}^{\infty} u_i \quad \text{for} \quad n = 1, 2, \ldots;$$

$$F(t) = 0 \quad \text{for} \quad t \geq 0; \quad F(t) = \sum_n (-1)^n nu_n \quad \text{for} \quad t < t_1;$$

$$F(t_n) = \sum_{i=n}^{\infty} (-1)^i iu_i \quad \text{for} \quad n = 1, 2, \ldots.$$

Finally, we define $F$ to be a linear function on each $(t_n, t_{n+1})$. Observe that for every integer $n > 0$ and every $u \in (u_{n+1}, u_n)$ we have $|F(d + u) - F(d)| \leq (n + 1)u_{n+1}$ for $d \geq t_{n+1}$ and $|F(d + u) - F(d)| \leq nu$ for $d < t_{n+1}$. Hence for every real number $d$

$$(++) \quad |F(d + u) - F(d)| \leq (n + 1)u \quad \text{for} \quad u \in (u_{n+1}, u_n).$$
More, observe that for \( u \geq u_1 \) and for every real number \( d \)

\[
|F(d + u) - F(d)| \leq u_1.
\]

Given two different real numbers \( w, w' \), if \( |w - w'| \geq u_1 \) then by

\[
\varphi(F(w) - F(w')) \leq \varphi(u_1) \leq c\varphi(|w - w'|) \quad \text{if } |w - w'| \in (u_{n+1}, u_n)
\]

then by \((++)\)

\[
\varphi(|F(w) - F(w')|) \leq \varphi((n + 1)|w - w'|) \leq c\varphi(|w - w'|),
\]

because \( u_n < v_n \). We have proved that \( F \in GL_\varphi \). Finally, for \( n = 1, 2, \ldots \)

we have \( t_n \in (-2, 2) \) and

\[
\frac{|F(t_{n+1}) - F(t_n)|}{|t_{n+1} - t_n|} = \frac{n u_n}{u_n} = n.
\]

Thus \( F \not\in GL \). So if \( GL_\varphi \subset GL \) then \( \varphi \) satisfies the condition \( (E) \).

\[\Box\]

**Theorem 3.** The inclusion \( GL \subset GL_\varphi \) holds if and only if \( \varphi \) satisfies the condition \( \Delta_2 \) for small \( u \).

**Proof.** First, assume that \( \varphi \) satisfies \( \Delta_2 \) for small \( u \) and \( F \in GL \). Given \( m > 0 \), there exists a constant \( C > 0 \) such that \( \varphi(|F(u) - F(v)|) \leq \varphi(C|u - v|) \)

for \( u, v \in (-m, m) \). Because \( \varphi \) satisfies \( \Delta_2 \) for small \( u \), there exists a constant \( K_m > 0 \) such that \( \varphi(Cw) \leq K_m \varphi(w) \) for \( 0 \leq w \leq 2m \) (cf. [2], 1.02). Thus,

\[
\varphi(|F(u) - F(v)|) \leq K_m \varphi(|u - v|) \quad \text{for } u, v \in (-m, m).
\]

It follows that \( F \in GL_\varphi \).

Conversely, suppose that \( \varphi \) does not satisfy the condition \( \Delta_2 \) for small \( u \). Then

\[
\limsup_{u \to 0^+} \frac{\varphi(2u)}{\varphi(u)} = \infty.
\]

and it is easy to see that for \( F(u) = 2u \) we have \( F \in GL \) and \( F \not\in GL_\varphi \).

\[\Box\]

The following result is just exactly a generalization of Theorem 2.B of [1].

**Corollary.** The identity \( GL = GL_\varphi \) holds if and only if \( \varphi \) satisfies the conditions \( (E) \) and \( \Delta_2 \) for small \( u \).

**References**


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