UNITARY REPRESENTATIONS OF LIE GROUPS AND GÅRDING'S INEQUALITY

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Abstract. We prove two versions of Gårding's inequality for strongly elliptic operators in the enveloping Lie algebra associated with a unitary representation of a Lie group. We then deduce a characterization of the differential structure of the representation in terms of the elliptic operators.

1. Introduction

Let $(\mathcal{H}, G, U)$ denote a continuous representation of the connected Lie group $G$ by unitary operators $U(g), g \in G$, on the Hilbert space $\mathcal{H}$. Fix a basis $a_1, \ldots, a_d$ of the Lie algebra $\mathfrak{g}$ of $G$ and let $A_1, \ldots, A_d$ denote the skew self-adjoint generators of the one-parameter subgroups $t \in \mathbb{R} \mapsto U(e^{-t a_i})$. If $\alpha = (\alpha_1, \ldots, \alpha_d)$, with $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, we define $A^\alpha = A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_d^{\alpha_d}$ and set $\mathcal{H}_\alpha = \bigcap_\alpha D(A^\alpha)$. It follows by standard reasoning that $\mathcal{H}_\alpha$ is norm dense in $\mathcal{H}$.

A form

$$a|\alpha| \leq m$$

where $C = C_1 \cdots C_d$, is defined to be strongly elliptic if $\text{Re}((-1)^{m/2} P_m(\xi)) > 0$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$, where $P_m$ denotes the principal part of $C_m$, i.e.,

$$P_m = \sum_{\alpha : |\alpha| = m} c_\alpha \xi^\alpha.$$

Equivalently, $C_m$ is strongly elliptic if there is a $p > 0$ such that

$$\text{Re}((-1)^{m/2} P_m(\xi)) \geq p|\xi|^m$$

for all $\xi \in \mathbb{R}^d$. The largest value $p_m(= p_m(\xi))$ of $p$ for which this estimate is valid is called the ellipticity constant of $C_m$. Note that the strong ellipticity condition implies automatically that $m$ is even.

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Next we associate with each unitary representation \((\mathcal{H}, G, U)\), each basis 
\[a_1, \ldots, a_d\], of \(g\), and each strongly elliptic form \(C_m\), an operator

\[A_m(c) = \sum_{\alpha, |\alpha| \leq m} c_\alpha A^\alpha\]

with domain \(D(A_m) = \mathcal{H}_\infty\). We refer to the \(A_m\) as strongly elliptic operators.

The simplest example is the Laplacian

\[\Delta = -\sum_{i=1}^{d} A_i^2\]

corresponding to the form \(\xi \mapsto -|\xi|^2\).

**Theorem 1.1.** Let \(A_m(c)\) be strongly elliptic. For each \(p \in (0, p_m(c))\) there is a \(q \geq 0\) such that

\[(1.1) \quad \Re(x, A_m(c)x) \geq p(x, \Delta^{m/2}x) - q(x, x)\]

for all \(x \in \mathcal{H}_\infty\). Moreover, \(q\) can be chosen independently of the particular unitary representation.

An alternative, weaker version of the theorem can be stated in terms of the \(C^n\)-norms of the representation. These are defined by \(\|x\|_0 = \|x\|\) and

\[\|x\|_n = \sup_{0 \leq i \leq d} \|A_i x\|_{n-1},\]

where \(A_0 = I\). The \(C^n\)-subspace

\[\mathcal{H}_n = \bigcap_{|\alpha| \leq n} D(A^\alpha)\]

is a Banach space with respect to \(\|\cdot\|_n\), and \(\mathcal{H}_\infty\) is \(\|\cdot\|_n\)-dense in \(\mathcal{H}_n\) (see, for example, [G]).

**Theorem 1.2.** Let \(A_m(c)\) be strongly elliptic. For each \(p' \in (0, p_m(c))\) there is a \(q' \geq 0\) such that

\[(1.2) \quad \Re(x, A_m(c)x) \geq p'||x||^2_{m/2} - q'||x||^2\]

for all \(x \in \mathcal{H}_\infty\). Moreover, \(q\) can be chosen independently of the particular unitary representation.

It was established by R. Goodman [G] that \(\mathcal{H}_n = D(\Delta^{n/2})\) and the norm \(\|\cdot\|_n\) on \(\mathcal{H}_n\) is equivalent to the graph norm \(x \mapsto \|\Delta^{n/2} x\| + \|x\|\). But more recently ([R1], [R2]) the following more precise estimates have been obtained; for each \(n = 1, 2, \ldots\) and \(\epsilon > 0\) there is a \(c_n(\epsilon)\) such that

\[\|x\|_n \leq (1 + \epsilon)\|\Delta^{n/2} x\| + c_n(\epsilon)\|x\|\]

for all \(x \in \mathcal{H}_n\). Therefore Theorem 1.2 follows easily from Theorem 1.1. Next we sketch the proof of the latter result. It is essentially a consequence of the work of Langlands [L1], [L2].
2. Outline of the proof of Theorem 1.1

If $C_m$ is a strongly elliptic form with ellipticity constant $p_m$ and $p \in (0, p_m)$ then the form $C'_m$ defined by

$$C'_m(\xi) = C_m(\xi) - p(-|\xi|^2)^{m/2}$$

is also strongly elliptic, with ellipticity constant $p'_m = p_m - p$. Since $A_m(c') = A_m(c) - pA^{m/2}$ the inequality (1.1) is equivalent to the lower semiboundedness property

$$\text{Re}(x, A_m(c')x) \geq -q(x, x).$$

Therefore to prove Theorem 1.1 it suffices to prove that the real part of every strongly elliptic operator $A_m(c)$ is lower semibounded.

Let $A^\dag_m$ denote the formal adjoint of $A_m$ on $\mathcal{F}^\infty$, i.e.,

$$A^\dag_m = \sum_{a : |\alpha| \leq m} \bar{c}_\alpha (-1)^{|\alpha|} A^\alpha$$

on $\mathcal{F}^\infty$ where $A^{\alpha*} = A^\alpha_1 \cdots A^\alpha_d$. Then $R_m = (A_m + A^\dag_m)/2$ is a symmetric operator on $\mathcal{F}^\infty$. But it follows from the structure relations of $\mathfrak{g}$ that $A^{\alpha*} = A^\alpha$ modulo lower order terms, i.e.,

$$A^{\alpha*} = A^\alpha + \sum_{\beta : |\beta| < |\alpha|} c_{\alpha, \beta} A^\beta,$$

where the $c_{\alpha, \beta}$ are polynomials in the structure constants. Therefore $R_m$ is a strongly elliptic operator associated with a form $C'_m$ whose principal part $P'_m$ is given by

$$P'_m(\xi) = \sum_{\alpha : |\alpha| = m} (\text{Re} c_\alpha) \xi^\alpha.$$

Now it follows from Langlands' [L1] first theorem that $R_m$ is essentially self-adjoint, and from his second theorem that the self-adjoint closure $\overline{R}_m$ generates a continuous semigroup, which is automatically self-adjoint. But then $\overline{R}_m$ is lower semibounded by spectral theory. Hence there is a $q \geq 0$ such that

$$\text{Re}(x, A_m x) = (x, R_m x) \geq -q(x, x)$$

for all $x \in \mathcal{F}^\infty$. Therefore (1.1) follows from the previous reasoning.

Finally, Langlands' [L1] third theorem establishes that the semigroup $S$ generated by $\overline{R}_m$ has a representation independent kernel, i.e.,

$$S_t = \int_G dgp_t(g)U(g),$$

where $dg$ denotes the left invariant Haar measure and $p_t \in L_1(G; dg)$. Since

$$e^{qt} \leq \int_G dgp_t(g)|$$
one can then choose \( q \) to be independent of the particular unitary representation. This completes the outline of the proof.

3. Differential structure

Let \( A_m = A_m(c) \) be a strongly elliptic operator with formal adjoint \( A_m^\dagger \) and define

\[
B_{2m} = A_m^\dagger A_m = \sum_{\alpha : |\alpha| \leq m} \sum_{\beta : |\beta| \leq m} \bar{c}_\alpha c_\beta (-1)^{|\alpha|} A_\alpha^\dagger A_\beta
\]

on \( \mathcal{H}_\infty \). Since \( A_\alpha^\dagger A_\beta = A_{\alpha + \beta} \) modulo lower-order terms, it follows that \( B_{2m} \) is a strongly elliptic operator. Moreover, if \( P_m \) denotes the principal part of the form \( C_m \) associated with \( A_m \) and \( P_{2m}' \) the principal part of the form \( C_{2m}' \) associated with \( B_{2m} \) then

\[
P_{2m}'(\xi) = |P_m(\xi)|^2 \geq (\text{Re} P_m(\xi))^2.
\]

Therefore one has the inequality \( p_{2m}' \geq p_m^2 \) for the ellipticity constants, with equality whenever the principal part of \( C_m \) is real. Now applying Theorem 1.2 to \( B_{2m} \) one deduces the following.

**Corollary 3.1.** Let \( A_m(c) \) be strongly elliptic. For each \( p \in \langle 0, p_m(c) \rangle \) there is a \( q > 0 \), independent of the representation, such that

\[
\|x\|_m \leq \left(\frac{1}{p}\right)\|A_m(c)x\| + q\|x\|
\]

for all \( x \in \mathcal{H}_\infty \). Consequently \( A_m(c) \) is closed on \( \mathcal{H}_m \) and the \( C^m \)-norm \( \|\cdot\|_m \) is equivalent to the graph \( x \mapsto \|A_m(c)x\| + \|x\| \).

**Proof.** Replacing \( A_m(c) \) by \( B_{2m} = A_m(c)^\dagger A_m(c) \) in (1.2) one finds for each \( p' \in \langle 0, p_{2m}' \rangle \) a \( q' \geq 0 \) such that

\[
p'\|x\|_m^2 \leq \|A_m(c)x\|^2 + q'\|x\|^2
\]

for all \( x \in \mathcal{H}_\infty \). But \( p_{2m}' \geq p_m^2 \). Thus if \( p \in \langle 0, p_m \rangle \) and \( p' = p^2 \),

\[
\|x\|_m^2 \leq \left(\frac{1}{p^2}\right)\|A_m(c)x\|^2 + q'\|x\|^2.
\]

Then (3.1) follows by elementary reasoning. But

\[
\|A_m(c)x\| \leq \left[ \sum_{\alpha : |\alpha| \leq m} |c_\alpha|^2 \right] \|x\|_m
\]

and since \( \mathcal{H}_\infty \) is \( \|\cdot\|_m \)-dense in \( \mathcal{H}_m \) one immediately deduces the last statement of the corollary from (3.1) and (3.2).

Finally the foregoing reasoning extends to higher-order products. If \( A_{m_1}, \ldots, A_{m_n} \) are all strongly elliptic and \( m = m_1 + \cdots + m_n \) then

\[
B_{2m} = (A_{m_n}^\dagger \cdots A_{m_1}^\dagger)(A_{m_1} \cdots A_{m_n})
\]
is a strongly elliptic operator of order $2m$. Hence the same arguments show
that the $C^m$-norm is equivalent to the graph norm
\[ x \mapsto \|A_{m_1} \cdots A_{m_n}x\| + \|x\|. \]

Detailed proofs of these results will appear in [R2].

4. Semigroups bounds

The closure $\overline{A_m}$ of each strongly elliptic operator $A_m$ generates a strongly
continuous semigroup $S$ holomorphic in a sector $\Delta_m(\varphi) = \{z \in \mathbb{C}; \Re z > 0, |\Arg z| < \varphi\}$ by Langlands' second theorem. Then if $\vartheta \in [0, \varphi)$ it follows by
general theory that there exist $M_\vartheta \geq 1$ and $\omega_\vartheta \geq 0$ such that
\[ \|S_z\| \leq M_\vartheta e^{\omega_\vartheta |z|} \]
whenever $\Re z \geq 0$ and $|\Arg z| \leq \vartheta$. But the Gårding inequalities allow one to
infer that $M_\vartheta = 1$, at least for small $\vartheta$.

Corollary 4.1. Let $C_m$ be a strongly elliptic form with ellipticity constant $p_m$, define
\[ q_m = \sum_{\alpha, |\alpha| = m} |\text{Im } c_\alpha|, \]
and $\varphi_m = \tan^{-1} p_m/q_m$. Further let $S$ denote the holomorphic semigroup generated by $\overline{A_m(c)}$.

If $\vartheta \in [0, \varphi_m)$ then there is an $\omega_\vartheta \geq 0$ such that
\[ \|S_z\| \leq e^{\omega_\vartheta |z|} \]
for all $z \in \mathbb{C}$ with $\Re z > 0$, and $|\Arg z| \leq \vartheta$.

Proof. First, by Langlands’ estimates [L2], or by [R2], the semigroup $S$ is
holomorphic in the sector $\Delta_m(\varphi_m)$, and possibly in a larger sector. Thus if
$z \in \mathbb{C}$ with $\Re z > 0$ and $|\Arg z| \leq \vartheta$ then $zA_m$ generates a continuous
semigroup. But
\[ \Re(x, zA_mx) = (\Re z)(\Re(x, A_mx) - (\text{Im } z)\text{Im}(x, A_mx) \geq \Re z \Re(x, A_mx) - |\text{Im } z|q_m\|x\|^2_{m/2} \]
\[ - |\text{Im } z|r_m\|x\|_{m/2} \cdot \|x\|_{m/2-1}, \]
where
\[ r_m = \sum_{\alpha, |\alpha| \leq m} |\text{Im } c_\alpha|. \]

Now for each $\delta > 0$,
\[ \|x\|_{m/2} \cdot \|x\|_{m/2-1} \leq \delta \|x\|^2_{m/2} + (1/4\delta)\|x\|^2_{m/2-1}. \]

Moreover, for each $\sigma > 0$ there is a $k_\sigma > 0$ such that
\[ \|x\|^2_{m/2-1} \leq \sigma \|x\|^2_{m/2} + k_\sigma \|x\|^2. \]
Hence for each \( \epsilon > 0 \) there is a \( c_\epsilon > 0 \) such that

\[
\text{Re}(x, z A_m x) = (\text{Re} z) \text{Re}(x, A_m x) - |\text{Im} z| (q_m + \epsilon) \|x\|^2 - |\text{Im} z| c_\epsilon \|x\|^2.
\]

Now we can use the second form of Gårding's inequality (1.2) to deduce that for each \( p' \in (0, p_m) \) there is a \( q' > 0 \) such that

\[
\text{Re}(x, z A_m x) \geq (\text{Re} z) \text{Re}(x, A_m x) \left(1 - \frac{|\text{Im} z|}{\text{Re} z} \frac{q_m + \epsilon}{p'}\right) - |\text{Im} z| \|x\|^2 \left(q_m q' (q_m + \epsilon) + c_\epsilon \right).
\]

But by choosing \( p' \) close to \( p_m \) and \( \epsilon \) small, one can assure that \( (1 - (q_m + \epsilon)|\text{Im} z|/p' \text{Re} z) > 0 \). Then by another application of Gårding's inequality there is a \( q > 0 \) such that \( \text{Re}(x, A_m x) \geq -q \|x\|^2 \). Therefore

\[
\text{Re}(x, (z/|z|) A_m x) \geq -\omega_\theta \|x\|^2
\]

with \( \omega_\theta = q(1 - ((q_m + \epsilon)/p') \tan \theta) + q' q_m/p' + c_\epsilon \).

Finally,

\[
\frac{d}{d|z|} |S_z x|^2 e^{-2\omega_\theta |z|} = -\text{Re}(S_z x, ((z/|z|) A_m + \omega_\theta I) S_z x) e^{-2\omega_\theta |z|} \leq 0.
\]

Therefore, by integration,

\[
|S_z x| \leq e^{\omega_\theta |z|} \|x\|
\]

for all \( z \in \mathbb{C} \) with \( \text{Re} z \geq 0 \) and \( |\text{Arg} z| \leq \theta \).

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