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CAN THE WEYL ALGEBRA BE A FIXED RING?

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ABSTRACT. If a finite soluble group acts as automorphisms of a domain, then the invariant subring is not isomorphic to the first Weyl algebra $\mathbf{C}[t, d/dt]$.

Let $R = \mathbf{C}[t, d/dt]$ be the first Weyl algebra. We prove the following result.

Theorem. *Let G be a finite solvable group. Let $S \supset R$ be a $\mathbf{C}$-algebra such that*
(a) *S_R and ${}_RS$ are finitely generated,*
(b) *S is a domain,*
(c) *G acts as automorphisms of S, and $S^G = R$.*
Then $S = R$.

We will prove a rather more general result, from which the theorem follows. The original proof was improved by comments of T. J. Hodges. I would like to thank him for his interest, and for allowing his improvements to be included here.

Let B be an R-R-bimodule. We call B an *invertible bimodule*, if there exists another bimodule, C say, such that $B \otimes_R C$ is isomorphic to R as a bimodule. The invertible bimodules form a group under the operation of tensor product $\otimes_R$; this group is called the Picard group, denoted $\mathrm{Pic}(R)$. If σ, $\tau \in \mathrm{Aut}(R)$ are $\mathbf{C}$-linear algebra automorphisms of R, then we write ${}_\sigma R_\tau$ for the invertible bimodule which is R as an abelian group, and for which the right R-module action is given by

$$b \cdot x = b\tau(x) \qquad \text{for } x \in R, b \in {}_\sigma R_\tau$$

and the left R-module action is given by

$$x \cdot b = \sigma(x)b \qquad \text{for } x \in R, b \in {}_\sigma R_\tau.$$

There is a map $\mathrm{Aut}(R) \to \mathrm{Pic}(R)$ given by $\sigma \mapsto {}_\sigma R_1$. This is a group homomorphism. A key point in our analysis is the following result of J. T. Stafford.

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Theorem [3, Corollary 4.5]. *The map* $\mathrm{Aut}(R) \to \mathrm{Pic}(R)$ *is an isomorphism.*

Hence if B is an invertible R-bimodule, there exists $e \in B$ and $\sigma \in \mathrm{Aut}(R)$ such that $x \cdot e = e \cdot \sigma(x)$ for all $x \in R$ (just take e to be the image of 1 under the isomorphism ${}_{\sigma}R_1 \to B$).

Proposition. *Let* $S \supset R$ *be a* $\mathbf{C}$*-algebra satisfying conditions* (a) *and* (b) *of the theorem. Then the only invertible* R*-bimodule contained in* S *is* R *itself.*

Proof. Let $B \subset S$ be an invertible bimodule. Choose $e \in B$ and $\sigma \in \mathrm{Aut}(R)$ such that $B = Re = eR$ and $x \cdot e = e \cdot \sigma(x)$ for all $x \in R$ (here $\cdot$ denotes multiplication in S). The multiplication in S is an R-bimodule map, so $B^n \cong_{\sigma^n} R_1$. If σ has infinite order (or equivalently, if B has infinite order in $\mathrm{Pic}\, R$), then all the bimodules B^n are non-isomorphic, and their sum in S would be direct. However, since S_R is finitely generated, S has finite length as an R-bimodule. Therefore, $\sigma^n = 1$ for some n. Hence for all $x \in R$, $xe^n = e^n \sigma^n(x) = e^n x$.

Therefore there is a surjective algebra homomorphism $R \otimes_{\mathbf{C}} \mathbf{C}[X] \to R[e^n]$, with $X \mapsto e^n$, where X is an indeterminate commuting with R. By [1, 4.5.1], the ideals of $R \otimes_{\mathbf{C}} \mathbf{C}[X]$ are of the form $R \otimes_{\mathbf{C}} I$ where I is an ideal of $\mathbf{C}[X]$. For $R[e^n] \cong R \otimes_{\mathbf{C}} \mathbf{C}[X]/R \otimes_{\mathbf{C}} I \cong R \otimes_{\mathbf{C}} \mathbf{C}[X]/I$ to be a domain it is necessary that $I = \langle X - \alpha \rangle$ for some $\alpha \in \mathbf{C}$. Thus $e^n = \alpha$. But $\mathbf{C}[e] \subset S$ is a domain, so $n = 1$. Therefore $B = R$. ■

If M is a left R-module, then the *rank* of M is the dimension of $\mathrm{Fract}\, R \otimes_R M$ as a left $\mathrm{Fract}\, R$-module. It is clear that an invertible bimodule is of rank 1.

Proof of the theorem. First we prove it for G abelian. In that case write $S = \bigoplus_{\chi} S_{\chi}$ where the sum is over the irreducible characters of G, and S_{χ} is the $\mathbf{C}G$-submodule of S which is the sum of the χ-isotypical components. Therefore $S_1 = R$, $S_{\chi} S_{\xi} = S_{\chi\xi}$, and each S_{χ} is an R-bimodule.

Suppose that $\chi\xi = 1$, and let $0 \neq a \in S_{\xi}$. Then $S_{\chi} a \subset R$, and is isomorphic to S_{χ} as a left R-module since S is a domain. In particular, S_{χ} is of rank 1 as a left R-module. Similarly, S_{χ} is of rank 1 as a right R-module. The multiplication map on S gives an R-bimodule homomorphism $S_{\chi} \otimes_R S_{\xi} \to S_{\chi} S_{\xi}$. The image is a non-zero subbimodule of R, hence equals R. Because all the ranks are 1, the map is injective. Therefore S_{χ} is an invertible bimodule. By the proposition, this forces $S_{\chi} = R$. Hence $S = R$ as required.

Now let G be any finite solvable group, and set $H = [G, G]$. Then there is an action of G/H as automorphisms of S^H, and $R = S^G = (S^H)^{G/H}$. But G/H is abelian, and the first part of the argument applied to S^H shows that $S^H = R$. Now by induction on $|G|$, the theorem follows. ■

Remarks. 1. It would be very nice to have the same result for an arbitrary finite group G, but a new idea is necessary. Not much is known about finitely generated R-bimodules which are not invertible, and that is probably a prerequisite.

2. I do not know of any domain $S \supsetneq R$ such that (a) and (b) hold. It would be very interesting to know whether or not such an S could exist. I expect not.

3. More generally I think it would be an interesting question to look at some other well understood non-commutative algebras, and ask if they can occur as the fixed ring of some reasonable extension ring. See [2] for an example concerning primitive factor rings of $U(\mathrm{sl}(2))$.

References

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