CAN THE WEYL ALGEBRA BE A FIXED RING?

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ABSTRACT. If a finite soluble group acts as automorphisms of a domain, then the invariant subring is not isomorphic to the first Weyl algebra $C[t,d/dt]$.

Let $R = C[t,d/dt]$ be the first Weyl algebra. We prove the following result.

**Theorem.** Let $G$ be a finite solvable group. Let $S \supseteq R$ be a $C$-algebra such that

(a) $S_R$ and $rS$ are finitely generated,
(b) $S$ is a domain,
(c) $G$ acts as automorphisms of $S$, and $S^G = R$.

Then $S = R$.

We will prove a rather more general result, from which the theorem follows. The original proof was improved by comments of T. J. Hodges. I would like to thank him for his interest, and for allowing his improvements to be included here.

Let $B$ be an $R$-$R$-bimodule. We call $B$ an invertible bimodule, if there exists another bimodule, $C$ say, such that $B \otimes_R C$ is isomorphic to $R$ as a bimodule. The invertible bimodules form a group under the operation of tensor product $\otimes_R$; this group is called the Picard group, denoted $\text{Pic}(R)$. If $\sigma, \tau \in \text{Aut}(R)$ are $C$-linear algebra automorphisms of $R$, then we write $\sigma R_\tau$ for the invertible bimodule which is $R$ as an abelian group, and for which the right $R$-module action is given by

$$b \cdot x = b\tau(x) \quad \text{for } x \in R, b \in \sigma R_\tau,$$

and the left $R$-module action is given by

$$x \cdot b = \sigma(x)b \quad \text{for } x \in R, b \in \sigma R_\tau.$$

There is a map $\text{Aut}(R) \to \text{Pic}(R)$ given by $\sigma \mapsto \sigma R_1$. This is a group homomorphism. A key point in our analysis is the following result of J. T. Stafford.
Theorem [3, Corollary 4.5]. The map Aut(\(R\)) \(\rightarrow\) Pic(\(R\)) is an isomorphism.

Hence if \(B\) is an invertible \(R\)-bimodule, there exists \(e \in B\) and \(\sigma \in \text{Aut}(R)\) such that \(x \cdot e = e \cdot \sigma(x)\) for all \(x \in R\) (just take \(e\) to be the image of 1 under the isomorphism \(\sigma_{R_1} \rightarrow B\)).

Proposition. Let \(S \supset R\) be a \(C\)-algebra satisfying conditions (a) and (b) of the theorem. Then the only invertible \(R\)-bimodule contained in \(S\) is \(R\) itself.

Proof. Let \(B \subset S\) be an invertible bimodule. Choose \(e \in B\) and \(\sigma \in \text{Aut}(R)\) such that \(B = Re = eR\) and \(x \cdot e = e \cdot \sigma(x)\) for all \(x \in R\) (here \(\cdot\) denotes multiplication in \(S\)). The multiplication in \(S\) is an \(R\)-bimodule map, so \(B^n \cong \sigma^n_{R_1}\). If \(\sigma\) has infinite order (or equivalently, if \(B\) has infinite order in Pic\(R\)), then all the bimodules \(B^n\) are non-isomorphic, and their sum in \(S\) would be direct. However, since \(S_R\) is finitely generated, \(S\) has finite length as an \(R\)-bimodule. Therefore, \(\sigma^n = 1\) for some \(n\). Hence for all \(x \in R\), \(xe^n = e^n\sigma^n(x) = e^n x\).

Therefore there is a surjective algebra homomorphism \(R \otimes_C C[X] \rightarrow R[e^n]\), with \(X \mapsto e^n\), where \(X\) is an indeterminate commuting with \(R\). By [1, 4.5.1], the ideals of \(R \otimes_C C[X]\) are of the form \(R \otimes_C I\) where \(I\) is an ideal of \(C[X]\). For \(R[e^n] \cong R \otimes_C C[X]/R \otimes_C I \cong R \otimes_C C[X]/I\) to be a domain it is necessary that \(I = \langle X - \alpha \rangle\) for some \(\alpha \in C\). Thus \(e^n = \alpha\). But \(C[e] \subset S\) is a domain, so \(n = 1\). Therefore \(B = R\).

If \(M\) is a left \(R\)-module, then the rank of \(M\) is the dimension of Fract \(R \otimes_R M\) as a left Fract \(R\)-module. It is clear that an invertible bimodule is of rank 1.

Proof of the theorem. First we prove it for \(G\) abelian. In that case write \(S = \bigoplus \chi S_\chi\), where the sum is over the irreducible characters of \(G\), and \(S_\chi\) is the \(CG\)-submodule of \(S\) which is the sum of the \(\chi\)-isotypical components. Therefore \(S_1 = R\), \(S_\chi S_\xi = S_{\chi\xi}\), and each \(S_\chi\) is an \(R\)-bimodule.

Suppose that \(\chi_\xi = 1\), and let \(0 \neq a \in S_\xi\). Then \(S_\chi a \subset R\), and is isomorphic to \(S_\chi\) as a left \(R\)-module since \(S\) is a domain. In particular, \(S_\chi\) is of rank 1 as a left \(R\)-module. Similarly, \(S_\chi\) is of rank 1 as a right \(R\)-module. The multiplication map on \(S\) gives an \(R\)-bimodule homomorphism \(S_\chi \otimes_R S_\xi \rightarrow S_\chi S_\xi\). The image is a non-zero subbimodule of \(R\), hence equals \(R\). Because all the ranks are 1, the map is injective. Therefore \(S_\chi\) is an invertible bimodule.

By the proposition, this forces \(S_\chi = R\). Hence \(S = R\) as required.

Now let \(G\) be any finite solvable group, and set \(H = [G, G]\). Then there is an action of \(G/H\) as automorphisms of \(S^H\), and \(R = S^G = (S^H)^{G/H}\). But \(G/H\) is abelian, and the first part of the argument applied to \(S^H\) shows that \(S^H = R\). Now by induction on \(|G|\), the theorem follows.

Remarks. 1. It would be very nice to have the same result for an arbitrary finite group \(G\), but a new idea is necessary. Not much is known about finitely generated \(R\)-bimodules which are not invertible, and that is probably a prerequisite.
2. I do not know of any domain $S \supsetneq R$ such that (a) and (b) hold. It would be very interesting to know whether or not such an $S$ could exist. I expect not.

3. More generally I think it would be an interesting question to look at some other well understood non-commutative algebras, and ask if they can occur as the fixed ring of some reasonable extension ring. See [2] for an example concerning primitive factor rings of $U(\text{sl}(2))$.

References


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