THE TORUS LEMMA ON CALIBRATIONS, EXTENDED

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Abstract. The whole face $G(\varphi)$ of $m$-planes calibrated by a torus $m$-form $\varphi$ is determined by the torus face $G_T(\varphi)$. Indeed, $G(\varphi)$ results from applying a new closure operation to $G_T(\varphi)$.

1. Introduction

Over the past ten years the theory of calibrations has illuminated the occurrence and structure of singularities in $m$-dimensional area-minimizing surfaces. This note gives an extension of a much-used lemma on calibrations, the Torus Lemma (cf. §3). Our observations bear on recent work of D. Nance [N, e.g. Corollary 3.8] and M. Messaoudene [Me].

For surveys on calibrations see [H1], [M1], [M2]. For basic concepts and definitions see [M3, §1, §2], [M4, Chapter 4], the original paper [HL], or the new text [H2].

2. Definitions

In addition to the standard dual Euclidean norms on the exterior algebra $\wedge^m \mathbb{R}^n$ and its dual $\wedge^m \mathbb{R}^n^*$, there is another important dual pair of norms, called mass and comass. The comass $\|\varphi\|^*$ of a form $\varphi \in \wedge^m \mathbb{R}^n$ is the maximum value of $\varphi$ on the Grassmannian $G(m, \mathbb{R}^n)$ of oriented unit $m$-planes through 0 in $\mathbb{R}^n$:

$$\|\varphi\|^* = \max \{\langle \xi, \varphi \rangle : \xi \in G(m, \mathbb{R}^n)\}.$$

A form $\varphi$ normalized to have comass 1 is called a calibration. The face $G(\varphi)$ of a calibration $\varphi$ consists of its maximum points in the Grassmannian:

$$G(\varphi) = \{\xi \in G(m, \mathbb{R}^n) : \langle \xi, \varphi \rangle = 1\}.$$

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The mass is the dual norm on $m$-vectors $\xi$ in $\bigwedge^m \mathbb{R}^n$:

$$
\|\xi\| = \max\{\langle \xi, \phi \rangle : \|\phi\|^* = 1\}
= \min\left\{ \sum a_j : \xi = \sum a_j \xi_j, \xi_j \in G(m, \mathbb{R}^n), a_j > 0 \right\}.
$$

A calibration $\phi$ for which the maximum of $\langle \xi, \phi \rangle$ is attained is said to calibrate $\xi$. An expression $\xi = \sum a_j \xi_j$ for which the minimum is attained is called a mass decomposition for $\xi$. A calibration $\phi$ calibrates $\xi = \sum a_j \xi_j$ if and only if all the $\xi_j$ lie in the face $G(\phi)$.

In the case that $n = 2m$ and $\mathbb{R}^n = (\mathbb{R}^2)^m$ consider the $m$-dimensional torus

$$
T = (S^1)^m = (G(1, \mathbb{R}^2))^m \subset G(m, \mathbb{R}^2m) \subset \bigwedge^m \mathbb{R}^{2m}.
$$

The elements of $T$ are called torus planes. Let $T_s$ denote the span of $T$ in $\bigwedge^m \mathbb{R}^{2m}$. The torus span $T_s$ can also be described as the tensor product $T_s = \bigotimes_{i=1}^m \bigwedge^1 \mathbb{R}^2$. Elements of $T_s$ are called torus $m$-vectors. Similarly the elements of the dualspace $T^*$ are called torus forms. The intersection of the face $G(\phi)$ of any calibration $\phi$ with the torus $T$ is called the torus face $G_T(\phi)$.

3. The Torus Lemma

The Torus Lemma ([M5, Lemma 4], cf. [DHM, §4]) says that a torus calibration attains its maximum value of 1 on the torus. Equivalently,

$$
\left\{ \phi \in T^*_s : \max_{\xi \in T} \langle \xi, \phi \rangle = 1 \right\} = \{ \phi \in T^*_s : \|\phi\|^* = 1 \}.
$$

We observe a few useful consequences.

1. The unit mass ball intersects the torus span in the convex hull of $T$.
2. A torus $m$-vector is calibrated by a torus calibration [Me, 4.4.2].
3. A torus $m$-vector has a mass decomposition in terms of torus $m$-planes. (This generalizes [Me, 5.2.8, 5.4.3].)
4. Let $\phi$ be a torus calibration. An $m$-plane $\xi$ belongs to $G(\phi)$ if and only if its projection $P\xi$ onto the torus span $T_s$ is a convex combination of $G_T(\phi)$.

Consequence (4) includes the new observation that the face $G(\phi)$ of a torus form $\phi$ is determined by the torus face $G_T(\phi)$. This fact is applied by D. Nance [N, e.g. Corollary 3.8]. Theorem 6 will exhibit this fact in another way.

The consequences follow immediately from (*) . For example, we will verify (4). Since $\phi$ is a torus calibration, $\phi(\xi) = \phi(P\xi)$. Clearly if $P\xi$ is a convex combination of $G_T(\phi)$, then $\phi(\xi) = \phi(P\xi) = 1$, so that $\xi \in G(\phi)$. On the other hand, suppose $\xi \in G(\phi)$, so that $\phi(P\xi) = \phi(\xi) = 1$. For any other torus calibration $\phi'$, $\phi'(P\xi) = \phi'(\xi) \leq 1$. It follows by elementary convex geometry from the characterization (*) of torus calibrations as

$$
\left\{ \phi \in T^*_s : \max_{\xi \in T} \langle \xi, \phi \rangle = 1 \right\}
$$

that $P\xi$ is a convex combination of $G_T(\phi)$. 

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4. Definitions

We call $A \subset G(m, \mathbb{R}^n)$ a CP$^1$ if for some orthonormal basis $e_1, \ldots, e_n$ for \(\mathbb{R}^n\) and the complex structure \(Je_1 = e_3, Je_2 = e_4\) on \(\mathbb{R}^4 = \text{span}\{e_1, e_2, e_3, e_4\}\),

\[A = \{\text{the complex lines in } \mathbb{R}^4\} \wedge e_5 \wedge \cdots \wedge e_{m+2}.
\]

Let $B \subset G(m, \mathbb{R}^n)$. We define the CP$^1$-closure $C(B)$ of $B$ as the smallest subset of $G(m, \mathbb{R}^n)$ containing $B$ such that whenever two points of a CP$^1$ lie in $C(B)$, the whole CP$^1$ lies in $C(B)$.

**Proposition 5 ([HM, Corollary 4.7]).** Let $G(\varphi)$ be a face of $G(m, \mathbb{R}^n)$. If two points of a CP$^1$ lie in $G(\varphi)$, then the whole CP$^1$ lies in $G(\varphi)$.

**Theorem 6.** Let $\varphi \in \mathbb{R}^{2m*}$ be a torus calibration. Then the face $G(\varphi)$ is the CP$^1$-closure of the torus face $G_T(\varphi)$:

\[G(\varphi) = C(G_T(\varphi)).\]

**Remarks.** This theorem subsumes both the Torus Lemma, which just says that $G_T(\varphi) \neq \emptyset$, and our new observation that the entire face of a torus calibration is determined by its torus face (cf. §3).

**Proof.** By Proposition 5, $G(\varphi) \supset C(G_T(\varphi))$. We prove the opposite inclusion by induction on $m$. The result is trivial for $m = 1$. Suppose $\varphi \in \bigotimes_{j=1}^{m+1}(\mathbb{A}^1 \mathbb{R}^{2*r}) \subset (\mathbb{A}^1 \mathbb{R}^{2*r}) \otimes (\mathbb{A}^m \mathbb{R}^{2m*})$. Let $\xi \in G(\varphi)$. It is not hard to show ([HL, Lemma II.7.5]) that there are orthonormal bases $e_1, e_2$ for $\mathbb{R}^2$ and $f_1, \ldots, f_{2m}$ for $\mathbb{R}^{2m}$ and angles $\theta_1, \theta_2 \in [0, \pi/2]$ such that $\xi$ takes the form

\[\xi = (\cos \theta_1, \sin \theta_1, f_1) \wedge (\cos \theta_2, \sin \theta_2, f_2) \wedge f_3 \wedge \cdots \wedge f_{m+1}.
\]

Since $\varphi \in \mathbb{A}^1 \mathbb{R}^{2*} \otimes \mathbb{A}^m \mathbb{R}^{2m*}$,

\[\varphi(\xi) = a \cos \theta_1 + b \sin \theta_2 \cos \theta_2 \leq \sqrt{a^2 \cos^2 \theta_1 + b^2 \sin^2 \theta_2} \leq \max\{|a|, |b|\} \leq 1,
\]

where

\[a = \langle e_1 \wedge f_2 \wedge \cdots \wedge f_{m+1}, \varphi \rangle,\]

\[b = \langle f_1 \wedge e_2 \wedge \cdots \wedge f_{m+1}, \varphi \rangle.
\]

Hence, equality holds. Unless $a = b = 1$, it follows that $\{\theta_1, \theta_2\} = \{0, \pi/2\}$ and $\xi$ has a factor $e_1$ or $e_2$, say $e_1$. Thus $\xi = e_1 \wedge \zeta$, for some $\zeta \in G(m, \mathbb{R}^{2m})$. Since $e_1 \perp \varphi \in \bigotimes_{j=1}^{m+1} \mathbb{A}^1 \mathbb{R}^{2*r}$ and $\langle \zeta, e_1 \perp \varphi \rangle = \pm \langle e_1 \wedge \zeta, \varphi \rangle = \pm 1$, by induction $\pm \zeta$ lies in $C(G_T(e_1 \perp \varphi))$. Consequently $\xi = e_1 \wedge \zeta$ belongs to

\[\pm e_1 \wedge C(G_T(e_1 \perp \varphi)) \subset \pm C(e_1 \wedge G_T(e_1 \perp \varphi)) \subset C(G_T(\varphi))
\]
as desired.
If on the other hand $a = b = 1$, then $\theta_2 = \pi/2 - \theta_1$. Also, $e_1 \wedge f_2 \wedge \cdots \wedge f_{m+1}$ and $f_1 \wedge e_2 \wedge \cdots \wedge f_{m+1}$ both belong to $G(\varphi)$. As in the previous case, by induction both belong to $C(G_T(\varphi))$. In addition for the complex structure $Je_1 = f_2$, $Jf_1 = e_2$, both belong to the $\mathbb{CP}^1$.

$$A = \{\text{complex lines in span } \{e_1, f_2, e_2\} \wedge f_3 \wedge \cdots \wedge f_{m+1}\}.$$

Since $\theta_2 = \pi/2 - \theta_1$, $\xi$ also belongs to $A$. Therefore $\xi \in C(G_T(\varphi))$, as desired.

References


[H2] ____, Spinors and calibrations, manuscript.


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