ON SECOND-CATEGORY SETS

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Abstract. The existence of a measurable cardinal is equiconsistent to the existence of a second category set not decomposable into the union of uncountable many disjoint second category sets.

It is easy to show that every second category set of reals can be decomposed into the disjoint union of infinitely many disjoint second category sets. Ulam [11] proved that under CH even uncountably many such sets can be found (see also [5], [9]). It is clear from saturated ideal theory that if there exists a set without such a decomposition, then there exists a model with a measurable cardinal. Also, the cardinality of the counter-example set is smaller than continuum but weakly Mahlo (see [8], [10]). In this note we show that these sets may exist.

Theorem. If the existence of a measurable cardinal is consistent, then so is the existence of a second category set in \( \mathbb{R} \) that is not the disjoint union uncountably many second category sets.

Proof. Let \( V \) be a countable model with \( \kappa \) a measurable cardinal and \( I \) a \( \kappa \)-complete normal prime ideal on \( \kappa \).

Let \( P \) be the notion of forcing, adding \( \kappa \) Cohen reals side-by-side with finite supports. We denote the \( \alpha \)th Cohen real by \( r_\alpha \). For every \( A \in I \) in \( V^P \), we let \( Q(A) \) be the canonical notion of forcing, making \( \{r_\alpha : \alpha \in A\} \) first category. This is defined as follows. Enumerate the diadic intervals as \( \{I_\alpha : n < \omega\} \). \( q = (s, N, g, f_0, \ldots, f_n) \) is in \( Q(A) \), if \( s \in [A]^{<\omega} \), \( N < \omega \), \( f_\alpha \) is a function defined on \( \{I_0, \ldots, I_N\} \) with \( f_{\alpha}(I_i) \) a diadic subinterval of \( I_i \), \( g : s \to \{0,1,\ldots,n\} \), such that if \( \alpha \in s \), \( i = g(s) \), then \( r_\alpha \notin f_{\alpha}(I_i), t = 0,1,\ldots,N \). \( (s', N', g', f'_0, \ldots, f'_{n'}) \leq (s, N, g, f_0, \ldots, f_n) \) if \( s' \supseteq s \), \( N' \geq N \), \( g' \supseteq g \), \( f'_{i} \supseteq f_{i} \) \( (i \leq n) \). It is well known that \( Q(A) \) is ccc and as required. Clearly, \( Q(A) \) is in \( V[\{r_\alpha : \alpha \in A\}] \). By \( \Delta \)-system arguments, \( Q = X\{Q(A) : A \in I\} \) is ccc, and so \( P \ast Q \) is also ccc.

Our aim is to show that in \( V^P \), \( X = \{r_\alpha : \alpha < \kappa\} \) witnesses the theorem. Assume first that \( p \models "X = \bigcup\{Y_{\xi} : \xi < \omega_i\} \) is a decomposition". We put

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$A_\xi = \{ \alpha < \kappa : \exists p' \leq p, p' \forces \neg \forall n \in Y_\xi \}$ These sets are defined in $V$. If an $A_\xi$ is in $I$, then forcing by $Q(A_\xi)$, and therefore forcing by $Q$, makes $Y_\xi$ first category. Otherwise, every $A_\xi$ is of measure one, i.e., $\kappa - A_\xi \in I$. Select an $\alpha \in \cap \{ A_\xi : \xi < \omega_1 \}$. For every $\xi < \omega_1$, there is a $p_\xi \leq p$, forcing $r_\alpha \in Y_\xi$. As $p$ forces so that the $Y_\xi - s$ are disjoint, the $p_\xi - s$ are pairwise incompatible, which contradicts the fact that $P \ast Q$ is ccc.

We now show that $X$ remains a set of second category even in $V^{P . Q}$. We may assume that $1 \Vdash_{P \ast Q} \langle X \rangle$ is of first category", for otherwise we can find a model with an $X$ of second category, and we are done. By a well-known lemma of forcing theory, there are names $F_0, F_1, \ldots$ such that $1$ forces the $F_n - s$ to be closed, nowhere dense sets, which cover $X$. There are, in $V^P$, conditions $q_i \in Q$ that completely define the $F_n - s$ by ccc. There are, in $V$, sets $A_0, A_1, \ldots \in I$ such that $\text{supp}(q_i) \subseteq \{ A_0, A_1, \ldots \}$, again by ccc. Select an $\alpha \in \kappa - (A_0 \cup A_1 \cup \cdots)$. If $Q = X\{ Q(A_i) : i < \omega \}, \bigcup F_n$ is in $V^{P . Q'}$; so by absoluteness, $r_\alpha \in \bigcup F_n$ holds there. Let $H$ be $Q'$-generic, choose a real $y$ which is not in $\bigcup F_n$ or in any of the first-category sets, coded in $V' = V[\{ r_\beta : \beta \neq \alpha \}, H]$. This is possible because our models are countable. Then, as is well known, $y$ is Cohen-generic over $V'$ (see [4]). And in $V'[y]$ the set $Y = X - \{ r_\alpha \} \cup \{ y \}$ may have the same name as the one for $X$. $\bigcup F_n$ is also the same, but $y \notin \bigcup F_n$, a contradiction.

References