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SELF-ADJOINTNESS OF THE *-REPRESENTATION GENERATED BY THE SUM OF TWO POSITIVE LINEAR FUNCTIONALS

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ABSTRACT. Let ϕ and ψ be positive linear functionals on a $*$-algebra $\mathscr{A}$. When the closed $*$-representations π_ϕ and π_ψ of $\mathscr{A}$ generated by the GNS-construction for ϕ and ψ are self-adjoint, we shall show that $\pi_{\phi+\psi}$ is self-adjoint if and only if $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$; and there exists a self-adjoint extension ρ of $\pi_{\phi+\psi}$ such that $\rho(\mathscr{A})'_w = \pi_{\phi+\psi}(\mathscr{A})'_w$ if and only if $\pi_{\phi+\psi}(\mathscr{A})'_w$ is a von Neumann algebra.

1. INTRODUCTION

Self-adjointness and standardness of $*$-representations which are important for the study of unbounded $*$-representations have been investigated in [10, 13, 20, 24]. To study a Radon-Nikodym theorem and a Lebesgue decomposition theorem for positive linear functionals on $*$-algebras [9, 11] and an unbounded Tomita-Takesaki theory [11], we need to investigate the self-adjointness of the $*$-representation $\pi_{\phi+\psi}$ generated by the sum $\phi+\psi$ of positive linear functionals ϕ and ψ on a $*$-algebra. In [10] we have stated that if π_ϕ is self-adjoint and π_ψ is *bounded*, then $\pi_{\phi+\psi}$ is self-adjoint. But, it is pointed out by K. Schmüdgen and J. Friedrich that there is a gap in the proof of this theorem. In this paper we correct this theorem and investigate the self-adjointness of $\pi_{\phi+\psi}$ when π_ϕ and π_ψ are self-adjoint (*without the assumption of the boundedness of* π_ψ).

In §2 we show that when π_ϕ and π_ψ are self-adjoint (resp. standard), $\pi_{\phi+\psi}$ is self-adjoint (resp. standard) iff $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$; and $\pi_{\phi+\psi}$ has a self-adjoint (resp. standard) extension ρ of $\pi_{\phi+\psi}$ such that $\rho(\mathscr{A})'_w = \pi_{\phi+\psi}(\mathscr{A})'_w$ iff $\pi_{\phi+\psi}$ is a von Neumann algebra. Further, we show that if π_ϕ is self-adjoint (resp. standard) and ψ is ϕ-dominated, then $\pi_{\phi+\psi}$ is self-adjoint (resp. standard).

In §3 we investigate the self-adjointness and the standardness of $\pi_{\phi+\psi}$ in special cases (approximately admissible positive linear functionals, the polynomial algebras; the commutative algebras, etc.).

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2. Self-adjointness of $\pi_{\phi+\psi}$

Let ϕ be a positive linear functional on a $*$-algebra $\mathscr{A}$. The well-known GNS-construction yields a triple $(\pi_\phi, \lambda_\phi, \mathfrak{h}_\phi)$, where π_ϕ is a closed $*$-representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}_\phi$, and λ_ϕ is a linear map of $\mathscr{A}$ into the domain $\mathscr{D}(\pi_\phi)$ of π_ϕ satisfying $\lambda_\phi(\mathscr{A})$ is dense in $\mathscr{D}(\pi_\phi)$ with respect to the induced topology t_{π_ϕ}, and $\lambda_\phi(xy) = \pi_\phi(x)\lambda_\phi(y)$ for $x, y \in \mathscr{A}$ [9, 10, 20]. We now put

$$\mathscr{D}(\pi_\phi^*) = \bigcap_{x\in\mathscr{A}} \mathscr{D}(\pi_\phi(x)^*), \qquad \pi_\phi^*(x) = \pi_\phi(x^*)^*|_{\mathscr{D}(\pi_\phi^*)};$$

$$\mathscr{D}(\pi_\phi^{**}) = \bigcap_{x\in\mathscr{A}} \mathscr{D}(\pi_\phi^*(x)^*), \qquad \pi_\phi^{**}(x) = \pi_\phi^*(x^*)^*|_{\mathscr{D}(\pi_\phi^{**})}, x \in \mathscr{A}.$$

Then π_ϕ^* is a closed representation of $\mathscr{A}$, but it is not necessarily a $*$-representation; π_ϕ^{**} is a closed $*$-representation of $\mathscr{A}$ and $\pi_\phi \subsetneqq \pi_\phi^{**} \subsetneqq \pi_\phi^*$ in general [6, 13]. If $\pi_\phi(x) \in \mathscr{B}(\mathfrak{h}_\phi)$ for each $x \in \mathscr{A}$, then π_ϕ is said to be *bounded*, if $\overline{\pi_\phi(x^*)} = \pi_\phi(x)^*$ for each $x \in \mathscr{A}$, then π_ϕ is *standard*; and if $\pi_\phi^* = \pi_\phi$, then π_ϕ is *self-adjoint*; and if $\pi_\phi^* = \pi_\phi^{**}$, then π_ϕ is *algebraically self-adjoint.* We next define a weak commutant of π_ϕ by

$$\pi_\phi(\mathscr{A})'_w = \{C \in \mathscr{B}(\mathfrak{h}_\phi); (C\pi_\phi(x)\xi|\eta) = (C\xi|\pi_\phi(x^*)\eta) \\ \text{for each } x \in \mathscr{A} \text{ and } \xi, \eta \in \mathscr{D}(\pi_\phi)\}.$$

Then $\pi_\phi(\mathscr{A})'_w$ is a weakly closed $*$-invariant subspace of $\mathscr{B}(\mathfrak{h}_\phi)$, which equals $\pi_\phi^{**}(\mathscr{A})'_w$, but it is not necessarily an algebra [6, 13, 20]. For the relation of the self-adjointness of π_ϕ and the weak commutant $\pi_\phi(\mathscr{A})'_w$ we have the following result [14].

Lemma 2.1. *Consider the following statements*:

(1) π_ϕ *is standard.*
(2) π_ϕ *is self-adjoint.*
(3) π_ϕ *is algebraically self-adjoint.*
(4) $\pi_\phi(\mathscr{A})'_w\mathscr{D}(\pi_\phi) \subset \mathscr{D}(\pi_\phi)$.
(5) $\pi_\phi(\mathscr{A})'_w\mathscr{D}(\pi_\phi^{**}) \subset \mathscr{D}(\pi_\phi^{**})$.
(6) $\pi_\phi(\mathscr{A})'_w$ *is a von Neumann algebra.*

Then the following implications hold:

$$(1) \Rightarrow (2) \begin{matrix} \Rightarrow \\ \Rightarrow \end{matrix} \left.\begin{matrix} (3) \\ \\ (4) \end{matrix}\right\} \Rightarrow (5) \Rightarrow (6).$$

Any converse implications don't necessarily hold.

Though the statement (6) in Lemma 2.1 does not imply the statement (4), we have the following result [14, 22].

Lemma 2.2. *Suppose $\pi_\phi(\mathscr{A})'_w$ is a von Neumann algebra. We put*

$$\mathscr{D}((\pi_\phi)_1) = \left\{ \sum_{k=1}^{n} C_k\xi_k \,;\, C_k \in \pi_\phi(\mathscr{A})'_w \,, \xi_k \in \mathscr{D}(\pi_\phi) \,, \right.$$
$$\left. k = 1, 2, \dots, n\,;\; n \in \mathbf{N} \right\},$$

$$(\pi_\phi)_1(x)\left(\sum_{k=1}^{n} C_k\xi_k\right) = \sum_{k=1}^{n} C_k\pi_\phi(x)\xi_k \,,\; x \in \mathscr{A} \,,\; \sum_{k=1}^{n} C_k\xi_k \in \mathscr{D}((\pi_\phi)_1) \,.$$

Then $(\pi_\phi)_1$ is a $*$-representation of $\mathscr{A}$ whose closure $\hat{\pi}_\phi$ has the following properties:

$$\pi_\phi \subset \hat{\pi}_\phi \subset \pi_\phi^* \,, \tag{2.1}$$

$$\hat{\pi}_\phi(\mathscr{A})'_w = \pi_\phi(\mathscr{A})'_w \,, \tag{2.2}$$

$$\pi_\phi(\mathscr{A})'_w\mathscr{D}(\hat{\pi}_\phi) \subset \mathscr{D}(\hat{\pi}_\phi) \,. \tag{2.3}$$

Theorem 2.3. *Let $\mathscr{A}$ be a $*$-algebra and ϕ and ψ be positive linear functionals on $\mathscr{A}$. Suppose π_ϕ and π_ψ are self-adjoint (resp. standard). Consider the following statements*:

(1) $\pi_{\phi+\psi}$ *is self-adjoint* (*resp. standard*).
(2) $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$.
(3) *There exists a self-adjoint* (*resp. standard*) *extension ρ of $\pi_{\phi+\psi}$ such that $\rho(\mathscr{A})'_w = \pi_{\phi+\psi}(\mathscr{A})'_w$.*
(4) $\pi_{\phi+\psi}(\mathscr{A})'_w$ *is a von Neumann algebra.*

Then the following implications hold:

$$\begin{array}{c} (1) \Leftrightarrow (2) \\ \Downarrow \\ (3) \Leftrightarrow (4) \end{array} .$$

When the above (4) *holds, $\hat{\pi}_{\phi+\psi}$ is self-adjoint* (*resp. standard*) *and $\hat{\pi}_{\phi+\psi} = \widehat{\pi^{**}_{\phi+\psi}}$.*

Proof. We put

$$K_\phi\lambda_{\phi+\psi}(x) = \lambda_\phi(x) \,,$$
$$K_\psi\lambda_{\phi+\psi}(x) = \lambda_\psi(x) \,, \qquad x \in \mathscr{A} \,.$$

Then K_ϕ (resp. K_ψ) is extended to the bounded linear operator from $\mathfrak{h}_{\phi+\psi}$ into $\mathfrak{h}_\phi$ (resp. $\mathfrak{h}_\psi$). It is easily shown that

$$\begin{aligned} &K_\phi^*K_\phi \,,\; K_\psi^*K_\psi \in \pi_{\phi+\psi}(\mathscr{A})'_w \,, \\ &K_\phi^*K_\phi + K_\psi^*K_\psi = I \,. \end{aligned} \tag{2.5}$$

Suppose π_ϕ and π_ψ are self-adjoint.

$(3) \Rightarrow (4)$. This follows from Lemma 2.1.

$(4) \Rightarrow (3)$. It is sufficient to show $\hat{\pi}_{\phi+\psi}$ is self-adjoint. By (2.2) and (2.3) we have

$$\begin{aligned}(\lambda_\phi(y)|K_\phi\hat{\pi}_{\phi+\psi}(x)^*\eta) &= (K_\phi^*K_\phi\lambda_{\phi+\psi}(y)|\hat{\pi}_{\phi+\psi}(x)^*\eta)\\ &= (K_\phi^*K_\phi\pi_{\phi+\psi}(x)\lambda_{\phi+\psi}(y)|\eta)\\ &= (\pi_\phi(x)\lambda_\phi(y)|K_\phi\eta)\end{aligned}$$

and similarly,

$$(\lambda_\psi(y)|K_\psi\hat{\pi}_{\phi+\psi}(x)^*\eta) = (\pi_\psi(x)\lambda_\psi(y)|K_\psi\eta)$$

for each $x \in \mathscr{A}$, $\eta \in \mathscr{D}(\hat{\pi}_{\phi+\psi}(x)^*)$ and $y \in \mathscr{A}$. Hence, we have

$$K_\phi\eta \in \mathscr{D}(\pi_\phi(x)^*) \quad \text{and} \quad \pi_\phi(x)^*K_\phi\eta = K_\phi\hat{\pi}_{\phi+\psi}(x)^*\eta, \tag{2.6}$$

$$K_\psi\eta \in \mathscr{D}(\pi_\psi(x)^*) \quad \text{and} \quad \pi_\psi(x)^*K_\psi\eta = K_\psi\hat{\pi}_{\phi+\psi}(x)^*\eta \tag{2.7}$$

for each $x \in \mathscr{A}$ and $\eta \in \mathscr{D}(\hat{\pi}_{\phi+\psi}(x)^*)$, which implies, by the self-adjointness of π_ϕ and π_ψ,

$$K_\phi\mathscr{D}(\hat{\pi}^*_{\phi+\psi}) \subset \mathscr{D}(\pi_\phi) \quad \text{and} \quad K_\psi\mathscr{D}(\hat{\pi}^*_{\phi+\psi}) \subset \mathscr{D}(\pi_\psi). \tag{2.8}$$

For each $x \in \mathscr{A}$, $\eta \in \mathscr{D}(\hat{\pi}_{\phi+\psi}(x)^*)$ and $\xi \in \mathscr{D}(\hat{\pi}^*_{\phi+\psi})$ we have

$$\begin{aligned}(\hat{\pi}_{\phi+\psi}(x)^*\eta|\xi) &= (K_\phi\hat{\pi}_{\phi+\psi}(x)^*\eta|K_\phi\xi) + (K_\psi\hat{\pi}_{\phi+\psi}(x)^*\eta|K_\psi\xi) \qquad \text{(by 2.6, 2.7)}\\ &= (\pi_\phi(x)^*K_\phi\eta|K_\phi\xi) + (\pi_\psi(x)^*K_\psi\eta|K_\psi\xi) \qquad \text{(by 2.8)}\\ &= (\eta|K_\phi^*\pi_\phi(x)K_\phi\xi + K_\psi^*\pi_\psi(x)K_\psi\xi) \qquad \text{(by 2.6, 2.7)}\\ &= (\eta|K_\phi^*K_\phi\hat{\pi}^*_{\phi+\psi}(x)\xi + K_\psi^*K_\psi\hat{\pi}^*_{\phi+\psi}(x)\xi)\\ &= (\eta|\hat{\pi}^*_{\phi+\psi}(x)\xi),\end{aligned}$$

and so $\xi \in \bigcap_{x\in\mathscr{A}} \mathscr{D}(\overline{\hat{\pi}_{\phi+\psi}(x)}) = \mathscr{D}(\hat{\pi}_{\phi+\psi})$ for each $\xi \in \mathscr{D}(\hat{\pi}^*_{\phi+\psi})$, which means that $\hat{\pi}_{\phi+\psi}$ is self-adjoint. Since $\hat{\pi}_{\phi+\psi} \subset \widehat{\pi^{**}_{\phi+\psi}}$ and $\hat{\pi}_{\phi+\psi}$ is self-adjoint, it follows that

$$\hat{\pi}_{\phi+\psi} \subset \widehat{\pi^{**}_{\phi+\psi}} \subset \widehat{\pi^{**}_{\phi+\psi}}^{\,*} \subset \hat{\pi}^*_{\phi+\psi} = \hat{\pi}_{\phi+\psi}.$$

Hence we have $\hat{\pi}_{\phi+\psi} = \widehat{\pi^{**}_{\phi+\psi}}$.

$(1) \Rightarrow (2)$. This follows from Lemma 2.1.

$(2) \Rightarrow (1)$. Since $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$, it follows that $\pi_{\phi+\psi}(\mathscr{A})'_w$ is a von Neumann algebra and $\hat{\pi}_{\phi+\psi} = \pi_{\phi+\psi}$, which implies by the proof of $(4) \Rightarrow (3)$ that $\pi_{\phi+\psi}$ is self-adjoint.

Suppose π_ϕ and π_ψ are standard.

(4) $\Rightarrow$ (3). It follows from (2.6), (2.7) and the standardness of π_ϕ and π_ψ that

$$\begin{aligned}(\hat{\pi}_{\phi+\psi}(x)^*\eta|\xi) &= (\eta|K_\phi^*\overline{\pi_\phi(x)}K_\phi\xi + K_\psi^*\overline{\pi_\psi(x)}K_\psi\xi) \\ &= (\eta|\hat{\pi}_{\phi+\psi}(x^*)^*\xi)\end{aligned}$$

for each $x \in \mathscr{A}$, $\eta \in \mathscr{D}(\hat{\pi}_{\phi+\psi}(x)^*)$ and $\xi \in \mathscr{D}(\hat{\pi}_{\phi+\psi}(x^*)^*)$, which implies that $\overline{\hat{\pi}_{\phi+\psi}(x)} = \hat{\pi}_{\phi+\psi}(x^*)^*$ for each $x \in \mathscr{A}$; that is, $\hat{\pi}_{\phi+\psi}$ is standard. The implications (3) $\Rightarrow$ (4) and (1) $\Leftrightarrow$ (2) are trivial. This completes the proof.

Remark 2.4. (1) As a slight extension of Theorem 2.3 we have that if π_ϕ and π_ψ are self-adjoint (resp. standard) then $\pi_{\phi+\psi}$ is self-adjoint (resp. standard) iff $K_\phi^*K_\phi\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$ iff $K_\psi^*K_\psi\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$. The proof is similar to that of Theorem 2.3.
(2) In ([10, Theorem 3.1]) we have shown that if π_ϕ is self-adjoint and π_ψ is bounded, then $\pi_{\phi+\psi}$ is self-adjoint. But, $K_\psi\pi_{\phi+\psi}(x)^*$ is not necessarily closable, and so the statement (3.2) in this theorem does not hold. We can prove that $\pi_{\phi+\psi}$ is self-adjoint iff $K_\psi\pi_{\phi+\psi}(x)^*$ is closable for each $x \in \mathscr{A}$. In fact, suppose $K_\psi\pi_{\phi+\psi}(x)^*$ is closable for each $x \in \mathscr{A}$. It is clear that $K_\psi\pi_{\phi+\psi}(x^*) \subset \pi_\psi(x)^*K_\psi$ and $K_\psi\pi_{\phi+\psi}(x^*) \subset K_\psi\pi_{\phi+\psi}(x)^*$ for each $x \in \mathscr{A}$. Since $\pi_\psi(x)^*K_\psi$ is bounded and $K_\psi\pi_{\phi+\psi}(x)^*$ is closable, it follows that $\pi_\psi(x)^*K_\psi = \overline{K_\psi\pi_{\phi+\psi}(x)^*}$, which implies

$$\begin{aligned}(\pi_{\phi+\psi}(x)^*\xi|K_\psi^*K_\psi\eta) &= (K_\psi\pi_{\phi+\psi}(x)^*\xi|K_\psi\eta) \\ &= (\pi_\psi(x)^*K_\psi\xi|K_\psi\eta) \\ &= (\xi|K_\psi^*\pi_\psi(x)K_\psi\eta)\end{aligned}$$

for each $x \in \mathscr{A}$, $\xi \in \mathscr{D}(\pi_{\phi+\psi}(x)^*)$ and $\eta \in \mathscr{D}(\pi_{\phi+\psi})$. Hence, $K_\psi^*K_\psi\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$. By the above (1) $\pi_{\phi+\psi}$ is self-adjoint. The converse is easily shown.

In similar fashion to the proof of Theorem 2.3 we can prove the following:

Corollary 2.5. *Suppose π_ϕ and π_ψ are algebraically self-adjoint. Consider the following statements*:

(1) $\pi_{\phi+\psi}$ *is algebraically self-adjoint.*
(2) $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi^{**}_{\phi+\psi}) \subset \mathscr{D}(\pi^{**}_{\phi+\psi})$.
(2)′ $K_\phi^*K_\phi\mathscr{D}(\pi^{**}_{\phi+\psi}) \subset \mathscr{D}(\pi^{**}_{\phi+\psi})$.
(2)″ $K_\psi^*K_\psi\mathscr{D}(\pi^{**}_{\phi+\psi}) \subset \mathscr{D}(\pi^{**}_{\phi+\psi})$.
(3) *There exists a self-adjoint extension ρ of $\pi_{\phi+\psi}$ such that $\rho(\mathscr{A})'_w = \pi_{\phi+\psi}(\mathscr{A})'_w$.*
(4) $\pi_{\phi+\psi}(\mathscr{A})'_w$ *is a von Neumann algebra.*

Then the following implications hold:

$$\begin{array}{c}(1) \Leftrightarrow (2) \Leftrightarrow (2)' \Leftrightarrow (2)'' \\ \Downarrow \\ (3) \Leftrightarrow (4).\end{array}$$

When the above (4) *holds,* $\hat{\pi}_{\phi+\psi}$ *is algebraically self-adjoint.*

We next investigate the self-adjointness of $\pi_{\phi+\psi}$ when ϕ and ψ have some relation.

Proposition 2.6. *Suppose* π_ϕ *is self-adjoint* (*resp. algebraically self-adjoint, standard*) *and* ψ *is* ϕ*-dominated* (*that is,* $\exists \gamma > 0$; $\psi(x^*x) \le \gamma\phi(x^*x)$, $x \in \mathscr{A}$). *Then* $\pi_{\phi+\psi}$ *is self-adjoint* (*resp. algebraically self-adjoint, standard*).

Proof. Since ψ is ϕ-dominated, it follows from ([9, Theorem 3.2]) that there exists a positive operator H' in $\pi_\phi(\mathscr{A})'_w$ such that $\psi(y^*x) = (H'\lambda_\phi(x)|\lambda_\phi(y))$ for each x, $y \in \mathscr{A}$. Hence we have

$$(2.9) \qquad (\phi+\psi)(y^*x) = ((I+H')^{1/2}\lambda_\phi(x)|(I+H')^{1/2}\lambda_\phi(y)), \qquad \forall x, y \in \mathscr{A}.$$

Suppose π_ϕ is self-adjoint. Since $(I+H')^{1/2}\lambda_\phi(\mathscr{A})$ is a π_ϕ-invariant subspace of $\mathscr{D}(\pi_\phi)$, the restriction $\pi_\phi|_{(I+H')^{1/2}\lambda_\phi(\mathscr{A})}$ of π_ϕ is a $*$-representation of $\mathscr{A}$ whose closure is denoted by π_1. By (2.9) the $*$-representations $\pi_{\phi+\psi}$ and π_1 are unitarily equivalent. It is sufficient to show π_1 is self-adjoint. Since

$$\begin{aligned}(\pi_\phi(x)\lambda_\phi(y)|(I+H')^{1/2}\xi) &= (\pi_1(x)(I+H')^{1/2}\lambda_\phi(y)|\xi) \\ &= (\lambda_\phi(y)|(I+H')^{1/2}\pi_1(x)^*\xi)\end{aligned}$$

for each $x \in \mathscr{A}$, $\xi \in \mathscr{D}(\pi_1(x)^*)$ and $y \in \mathscr{A}$, we have

$$(2.10) \qquad \begin{gathered}(I+H')^{1/2}\xi \in \mathscr{D}(\pi_\phi(x)^*) \quad \text{and} \\ \pi_\phi(x)^*(I+H')^{1/2}\xi = (I+H')^{1/2}\pi_1(x)^*\xi\end{gathered}$$

for each $x \in \mathscr{A}$ and $\xi \in \mathscr{D}(\pi_1(x)^*)$. Since π_ϕ is self-adjoint, it follows that

$$(2.11) \qquad \eta \in \mathscr{D}(\pi_\phi) \quad \text{and} \quad \pi_\phi(x)(I+H')^{1/2}\eta = (I+H')^{1/2}\pi_1^*(x)\eta$$

for each $x \in \mathscr{A}$ and $\eta \in \mathscr{D}(\pi_1^*)$. By (2.10) and (2.11) we have

$$\begin{aligned}(\pi_1(x)^*\xi|\eta) &= ((I+H')^{1/2}\pi_1(x)^*\xi|(I+H')^{-1/2}\eta) \\ &= (\pi_\phi(x)^*(I+H')^{1/2}\xi|(I+H')^{-1/2}\eta) \\ &= (\xi|(I+H')^{1/2}\pi_\phi(x)(I+H')^{-1/2}\eta)\end{aligned}$$

for each $x \in \mathscr{A}$, $\xi \in \mathscr{D}(\pi_1(x)^*)$ and $\eta \in \mathscr{D}(\pi_1^*)$. Hence, $\eta \in \bigcap_{x\in\mathscr{A}} \mathscr{D}(\overline{\pi_1(x)})$ $= \mathscr{D}(\pi_1)$, and so π_1 is self-adjoint. Similarly we can show that if π_ϕ is algebraically self-adjoint (resp. standard) then $\pi_{\phi+\psi}$ is algebraically self-adjoint (resp. standard).

3. Special cases

In this section we investigate the self-adjointness and the standardness of $\pi_{\phi+\psi}$ in special cases.

I. *Approximately admissible positive linear functionals.* A positive linear functional ϕ on a $*$-algebra $\mathscr{A}$ is said to be *admissible* if for each $a \in \mathscr{A}$ there is a constant γ_a such that $|\phi(x^* a^* ax)| \leq \gamma_a \phi(x^* x)$ for all $x \in \mathscr{A}$. We note that ϕ is admissible iff π_ϕ is bounded. A positive linear functional ϕ on $\mathscr{A}$ is said to be *approximately admissible* if there exists a net $\{\phi_\alpha\}$ of admissible positive linear functionals ϕ_α on $\mathscr{A}$ such that $\phi_\alpha \leq \phi$ $(\phi_\alpha(x^*x) \leq \phi(x^*x)$, $\forall x \in \mathscr{A})$ for each α and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$; and ϕ is said to be *strictly unadmissible* if there does not exist any non-zero admissible positive linear functional ψ on $\mathscr{A}$ such that $\psi \leq \phi$.

Approximately admissible positive linear functional ϕ has been investigated in [8, 24] under the assumption that π_ϕ is self-adjoint. We here obtain the same results under the weaker assumption that $\pi_\phi(\mathscr{A})'_w$ is a von Neumann algebra. These results are proved in similar fashion to the proofs in [8, 24] by considering $\hat{\pi}_\phi$ instead of π_ϕ.

Proposition 3.1. (1) *Suppose ϕ and ψ are approximately admissible positive linear functionals on $\mathscr{A}$. Then $\phi + \psi$ is approximately admissible.*
(2) *Suppose ϕ is approximately admissible and $\pi_\phi(\mathscr{A})'_w$ is a von Neumann algebra. Then every positive linear functional ψ on $\mathscr{A}$ with $\psi \leq \phi$ is approximately admissible.*
(3) *Let ϕ be a positive linear functional on a $*$-algebra $\mathscr{A}$ with identity e. Suppose $\pi_\phi(\mathscr{A})'_w$ is a von Neumann algebra. Then there exists a unique projection E in $\pi_\phi(\mathscr{A})'_w$ such that ϕ_E is approximately admissible and ϕ_{I-E} is strictly unadmissible, where $\phi_E(x) = (E\lambda_\phi(x)|\lambda_\phi(e))$, $x \in \mathscr{A}$. Further, the above E equals $\sum_{\alpha\in\Lambda} E_\alpha$, where $\{E_\alpha\}$ is a set of mutually orthogonal projections in $\pi_\phi(\mathscr{A})'_w$ such that ϕ_{E_α} is admissible for each $\alpha \in \Lambda$.*

For the standardness of approximately admissible positive linear functionals ϕ Takesue has shown in ([24, Theorem 3.1]) that if π_ϕ is self-adjoint then π_ϕ is standard; and we have stated in ([10, Theorem 4.1]) that π_ϕ is standard for every approximately admissible positive linear functional ϕ. But, for the same reason as in ([10, Theorem 3.1]) the statement (4.1) in this theorem does not necessarily hold, and so we correct this result as follows:

Theorem 3.2. *Let ϕ be an approximately admissible positive linear functional on a $*$-algebra $\mathscr{A}$ with identity. Then the following statements hold*:
(1) *π_ϕ is standard iff π_ϕ is self-adjoint iff $\pi_\phi(\mathscr{A})'_w \mathscr{D}(\pi_\phi) \subset \mathscr{D}(\pi_\phi)$ iff π_ϕ is unitarily equivalent to a direct sum $\bigoplus_{\alpha\in\Lambda} \pi_\alpha$ [20] of a set $\{\pi_\alpha\}_{\alpha\in\Lambda}$ of bounded $*$-representations of $\mathscr{A}$.*
(2) *Suppose $\pi_\phi(\mathscr{A})'_w$ is a von Neumann algebra. Then $\hat{\pi}_\phi$ is standard and it is a direct sum $\bigoplus_{\alpha\in\Lambda} \pi_\alpha$ of a set $\{\pi_\alpha\}_{\alpha\in\Lambda}$ of bounded $*$-representations of $\mathscr{A}$.*

Corollary 3.3. *Let ϕ and ψ be approximately admissible. Suppose $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$ (resp. $\pi_{\phi+\psi}(\mathscr{A})'_w$ is a von Neumann algebra). Then $\pi_{\phi+\psi}$ (resp. $\hat{\pi}_{\phi+\psi}$) is standard.*

Proof. This follows from Proposition 3.1, (1) and Theorem 3.2.

II. *Polynomial algebras.* Let $\mathscr{P}(x_1, x_2, \dots, x_n)$ be a $*$-algebra generated by mutually commuting hermitian elements $\{x_1, x_2, \dots, x_n\}$ and ϕ, ψ be positive linear functionals on $\mathscr{P}(x_1, x_2, \dots, x_n)$.

Proposition 3.4. *Suppose π_ϕ and π_ψ are self-adjoint. Then the following statements hold*:

(1) *Suppose $\overline{\pi_{\phi+\psi}(x_k)}$ is self-adjoint for $k = 1, 2, \dots, n$. Then $\hat{\pi}_{\phi+\psi}$ is self-adjoint.*

(2) *Suppose $\overline{\pi_{\phi+\psi}(x_k^m)}$ is self-adjoint for $k = 1, 2, \dots, n$ and each $m \in \mathbf{N}$. Then the following statements hold*:

- (a) *$\pi_{\phi+\psi}$ is algebraically self-adjoint.*
- (b) *If the self-adjoint operators $\{\overline{\pi_{\phi+\psi}(x_k)};\ k = 1, 2, \dots, n\}$ have the mutually commuting spectral projections, then $\pi^{**}_{\phi+\psi}$ is standard and $(\pi_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w$ is commutative.*
- (c) *If $\mathscr{D}(\pi_{\phi+\psi}) = \bigcap_{m_1, \dots, m_n \in N} \mathscr{D}(\overline{\pi_{\phi+\psi}(x_1)}^{m_1}\overline{\pi_{\phi+\psi}(x_2)}^{m_2} \dots \overline{\pi_{\phi+\psi}(x_n)}^{m_n})$, then $\pi_{\phi+\psi}$ is self-adjoint.*

(3) *Suppose π_ϕ and π_ψ are standard. Then $\pi_{\phi+\psi}$ is standard iff $\pi_{\phi+\psi}$ is self-adjoint iff $\overline{\pi_{\phi+\psi}(x_k^m)}$ is self-adjoint for $k = 1, 2, \dots, n$ and each $m \in \mathbf{N}$ and $\mathscr{D}(\pi_{\phi+\psi}) = \bigcap_{m_1, \dots, m_n \in N} \mathscr{D}(\overline{\pi_{\phi+\psi}(x_1)}^{m_1}\overline{\pi_{\phi+\psi}(x_2)}^{m_2} \dots \overline{\pi_{\phi+\psi}(x_n)}^{m_n})$.*

Proof. (1) It is easily shown that

$$\pi_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w = \bigcap_{k=1}^{n} \pi_{\phi+\psi}(\mathscr{P}(x_k))'_w. \tag{3.1}$$

It hence follows from ([20, Lemma 3.2]) that $\pi_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w$ is a von Neumann algebra. By Theorem 2.3, $\hat{\pi}_{\phi+\psi}$ is self-adjoint.

(2) (a) Since $\pi^*_{\phi+\psi}$ is a representation of $\mathscr{A}$ and $\pi^*_{\phi+\psi}(x_k^m)^* = \pi_{\phi+\psi}(x_k^m)^* = \overline{\pi_{\phi+\psi}(x_k^m)}$ for $k = 1, 2, \dots, n$ and $m \in \mathbf{N}$, it follows that $\pi^*_{\phi+\psi}$ is a $*$-representation of $\mathscr{A}$, which means that $\pi_{\phi+\psi}$ is algebraically self-adjoint.

(b) It follows from (a) and ([13, Theorem 3.2]) that $\pi^{**}_{\phi+\psi}$ is standard. Since $\pi^{**}_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w = \pi_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w$, it follows from ([20, Theorem 7.1]) that $(\pi_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w)'$ is commutative.

(c) Take arbitrary $C \in \pi_{\phi+\psi}(\mathscr{P}(x_1, x_2, \dots, x_n))'_w$ and $\eta \in \mathscr{D}(\pi_{\phi+\psi})$. By (3.1) $C \in \pi_{\phi+\psi}(\mathscr{P}(x_k))'_w$ and

$$\eta \in \bigcap_{m_1, \dots, m_n \in N} \mathscr{D}(\overline{\pi_{\phi+\psi}(x_1)}^{m_1}\overline{\pi_{\phi+\psi}(x_2)}^{m_2} \dots \overline{\pi_{\phi+\psi}(x_n)}^{m_n}).$$

By ([13, Theorem 2.1]) we have

$$C\eta \in \bigcap_{m_1,\dots,m_n \in N} \mathscr{D}(\overline{\pi_{\phi+\psi}(x_1)}^{m_1}\overline{\pi_{\phi+\psi}(x_2)}^{m_2}\dots\overline{\pi_{\phi+\psi}(x_n)}^{m_n}).$$

Hence, $C\eta \in \mathscr{D}(\pi_{\phi+\psi})$. By Theorem 2.3, $\pi_{\phi+\psi}$ is self-adjoint.
(3) This follows from (c), Theorem 2.3 and ([13, Theorem 3.2]).

III. *Commutative algebras.* Let $\mathscr{A}$ be a commutative $*$-algebra and ϕ and ψ be positive linear functionals on $\mathscr{A}$. By ([20, Theorem 7.1]) and ([12, Theorem 3.1]) π_ϕ (resp. $\hat{\pi}_\phi$) is standard iff $\pi_\phi(\mathscr{A})'_w\mathscr{D}(\pi_\phi) \subset \mathscr{D}(\pi_\phi)$ (resp. $\pi_\phi(\mathscr{A})'_w$ is a von Neumann algebra) and $(\pi_\phi(\mathscr{A})'_w)'$ is commutative. Hence it follows from Theorem 2.3 that if π_ϕ and π_ψ are standard and $\pi_{\phi+\psi}(\mathscr{A})'_w\mathscr{D}(\pi_{\phi+\psi}) \subset \mathscr{D}(\pi_{\phi+\psi})$ (resp. $\pi_{\phi+\psi}(\mathscr{A})'_w$ is a von Neumann algebra) then $\pi_{\phi+\psi}$ (resp. $\hat{\pi}_{\phi+\psi}$) is standard and $(\pi_{\phi+\psi}(\mathscr{A})'_w)'$ is commutative.

IV. *Symmetric $*$-algebras.* Let $\mathscr{A}$ be a symmetric $*$-algebra (that is, x^*x is quasi-regular for each $x \in \mathscr{A}$). Then $\pi_{\phi+\psi}$ is standard for every positive linear functional ϕ and ψ on $\mathscr{A}$ [8].

V. *Locally convex $*$-algebras with dense bounded parts.* Let $\mathscr{A}$ be a locally convex $*$-algebra with jointly continuous multiplication. An element x of $\mathscr{A}$ is said to be *bounded* if for some non-zero $\lambda \in \mathbf{C}$, the set $\{(\lambda x)^n; n \in \mathbf{N}\}$ is bounded [1, 2, 4]. We denote by $\mathscr{A}_0$ the set of all bounded elements of $\mathscr{A}_0$.

Proposition 3.5. *Let ϕ and ψ be continuous positive linear functionals on $\mathscr{A}$.*

(1) *Suppose $\mathscr{A} = \mathscr{A}_0$. Then $\pi_{\phi+\psi}$ is bounded.*
(2) *Suppose $\mathscr{A}_0$ is dense in $\mathscr{A}$ and π_ϕ and π_ψ are self-adjoint. Then $\hat{\pi}_{\phi+\psi}$ is self-adjoint.*

Proof. In similar fashion to the proof of ([21, Theorem 4.5.2]) we can show that $\overline{\pi_f(\mathscr{A}_0)} \subset \mathscr{B}(\mathfrak{h}_f)$ for every continuous positive linear functional f on $\mathscr{A}$, which implies the statement (1). We show the statement (2). Since the map: $x \in \mathscr{A} \to (\phi+\psi)(x^*x) \in \mathbf{C}$ is continuous, it follows that $\pi_{\phi+\psi}(\mathscr{A})'_w = \pi_{\phi+\psi}(\mathscr{A}_0)'_w$, and so $\pi_{\phi+\psi}(\mathscr{A})'_w$ is a von Neumann algebra. By Theorem 2.3, $\hat{\pi}_{\phi+\psi}$ is self-adjoint.

References

1. G. R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. **15** (1965), 399–421.
2. ——, *On a class of locally convex algebras*, Proc. London Math. Soc. **17** (1967), 91–114.
3. S. J. Bhatt, *Representability of positive functionals on abstract star algebras without identity with applications to locally convex $*$-algebras*, Yokohama Math. J. **29** (1981), 7–16.
4. P. G. Dixon, *Generalized B^*-algebras*, Proc. London Math. Soc. **21** (1970), 693–715.
5. G. Epifanio and C. Trapani, *V^*-algebras: an extension of the concept of von Neumann algebras to unbounded operators*, J. Math. Phys. **25** (1984), 2633–2637.

6. S. P. Gudder and W. Scruggs, *Unbounded representations of $*$-algebras*, Pacific J. Math. **70** (1977), 369–382.
7. S. P. Gudder and R. L. Hudson, *A noncommutative probability theory*, Trans. Amer. Math. Soc. **245** (1978), 1–41.
8. A. Inoue, *Unbounded representations of symmetric $*$-algebras*, J. Math. Soc. Japan **29** (1977), 219–232.
9. ——, *A Radon-Nikodym theorem for positive linear functionals on $*$-algebras*, J. Operator Theory **10** (1983), 77–86.
10. ——, *Selfadjointness of $*$-representations generated by positive linear functionals*, Proc. Amer. Math. Soc. **93** (1985), 643–647.
11. ——, *An unbounded generalization of the Tomita-Takesaki theory* I, II, Publ. RIMS, Kyoto Univ. **22** (1986), 725–765; Publ. RIMS, Kyoto Univ. **23** (1987), 673–726.
12. ——, *An unbounded generalization of Hilbert algebras*, J. Math. Ann. Appl. **138** (1989), 559–568.
13. A. Inoue and K. Takesue, *Self-adjoint representations of polynomial algebras*, Trans. Amer. Math. Soc. **280** (1983), 393–400.
14. A. Inoue, H. Ueda and T. Yamauchi, *Commutants and bicommutants of algebras of unbounded operators*, J. Math. Phys. **28** (1987), 1–7.
15. P. Jørgensen and R. T. Moore, *Operator commutation relations*, D. Reidel, Boston-Dordrecht 1984.
16. G. Lassner, *Topological algebras of operators*, Rep. Math. Phys. **3** (1972), 279–293.
17. E. Nelson, *Analytic vectors*, Ann. Math. **70** (1959), 572–615.
18. A. E. Nussbaum, *On the integral representation of positive linear functionals*, Trans. Amer. Math. Soc. **128** (1967), 460–473.
19. J. D. Powell, *Representations of locally convex $*$-algebras*, Proc. Amer. Math. Soc. **44** (1974), 341–346.
20. R. T. Powers, *Self-adjoint algebras of unbounded operators*, I, II, Commun. Math. Phys. **21** (1971), 85–124; Trans. Amer. Math. Soc. **187** (1974), 261–293.
21. C. E. Richart, *General theory of Banach algebras*, Van Nostrand, Princeton, 1960.
22. K. Schmüdgen, *The order structure of topological $*$-algebras of unbounded operators*, Rep. Math. Phys. **7** (1975), 215–227.
23. ——, *Unbounded commutants and intertwining spaces of unbounded symmetric operators and $*$-representations*, J. Funct. Anal, **71** (1987), 47–68.
24. K. Takesue, *Standard representations induced by positive linear functionals*, Mem. Fac. Sci. Kyushu Univ. **37** (1983), 211–225.

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