FIRST EIGENVALUE OF THE LAPLACEAN AND TORSION OF PARALLELIZABLE RIEMANNIAN MANIFOLDS

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Abstract. Lower and upper bounds for the smallest positive eigenvalue of the Laplacean on a parallelizable Riemannian manifold are combined to obtain an explicit lower bound for the torsion of global orthonormal frame fields in terms of the diameter of the metric.

Introduction and results

Consider a compact, connected $n$-dimensional Riemannian manifold $M$ with metric tensor $g$. We assume that the tangent bundle $TM$ is trivial. Then there exist parallelizations $\omega: TM \to \mathbb{R}^n$ mapping $g$ into the euclidean metric on $\mathbb{R}^n$. Such $\omega$ will be called orthonormal. Equivalently, the vector fields $E_i$ defined by $\omega(E_i) = e_i = i$th standard basis vector of $\mathbb{R}^n$ form an orthonormal frame field on $M$.

The torsion $T$ of $\omega$ is defined by $T(X,Y) = \omega^{-1}(d\omega(X,Y))$ for vector fields $X$ and $Y$ on $M$. $T$ is the torsion tensor of the flat connection $D$ on $TM$ defined by $DE_i = 0$. It is well known that a compact Riemannian manifold admits a torsion-free orthonormal parallelization if and only if it is isometric to a flat torus.

In this note we study the relation between isometric invariants of a Riemannian manifold and the torsion of its orthonormal frame fields. Specifically, let $\kappa$ denote the maximum norm of $T$, i.e.,

$$\kappa = \max \{ \| T(X,Y) \| : X,Y \in T_x M, \| X \| = \| Y \| = 1, x \in M \}. $$

The smallest positive eigenvalue of the Laplace operator of $(M,g)$ on functions is denoted by $\lambda_1$.

Theorem 1. Let $(M,g)$ be a compact connected Riemannian manifold and $\omega: TM \to \mathbb{R}^n$ an orthonormal parallelization. If for some $p$ $(1 \leq p \leq n-1)$ the Betti number $b_p$ of $M$ satisfies $b_p > \binom{n}{p}$, then $\lambda_1 \leq 100n^8\kappa^2$.

Example. Let $M = \Gamma \backslash G$ be a nilmanifold. Then there exist a sequence of Riemannian metrics $\{g_\nu\}$ and a corresponding sequence $\{\omega_\nu\}$ of orthonormal...
parallelizations on \( M \) such that \( \kappa^2/\lambda_1 \) converges to zero as \( \nu \to \infty \). For \( g_\nu \) one can use the almost flat metrics constructed by Gromov and for \( \omega_\nu \) Maurer-Cartan forms (see [2]).

Theorem 1 can be combined with lower bounds on \( \lambda_1 \). Here we note the following consequence:

**Theorem 2.** Under the assumptions of Theorem 1 we have \( \kappa d \geq 50^{-n} \), where \( d \) is the diameter of \( (M, g) \).

The existence of some positive \( \varepsilon = \varepsilon(n) \) such that \( \kappa d \geq \varepsilon \) also follows from [1].

**The proof**

We use tensor notation and the summation convention for the computations. Components of tensors are taken with respect to the orthonormal frame \( \{E_i\} \) and coframe \( \{\omega^i\} \), where \( \omega = \omega^i \otimes e_i \). For example, \( T = T_{ij} \omega^i \otimes \omega^j \otimes E_k \). Since \( g_{ij} = \delta_{ij} \), we need not distinguish between upper and lower indices.

If \( \nabla \) denotes the Levi-Civitè connection of \( g \) then \( \nabla - D = S \), where \( S \) is given by \( S_{ijk} = \frac{1}{2} \left( -T^k_{ij} + T^k_{jk} - T^k_{ki} \right) \). We obtain the following expressions for the exterior derivative \( d \) and its formal adjoint \( \delta \) acting on a \( p \)-form \( \alpha \):

\[
(d\alpha)_{i_0 \cdots i_p} = \sum_{\nu=0}^{p} (-1)^\nu \alpha_{i_0 \cdots \tilde{i}_\nu \cdots i_p} + \sum_{\mu < \nu} (-1)^{\mu + \nu} T^i_{\mu i_\nu} \alpha_{j_0 \cdots \tilde{i}_\nu \cdots \tilde{i}_\mu \cdots i_p} \\
(\delta \alpha)_{i_2 \cdots i_p} = -\alpha_{i_2 \cdots i_p, i} + T^i_{ij} \alpha_{k i_2 \cdots i_p} + \sum_{\nu=2}^{p} S_{i_\nu k} \alpha_{i_2 \cdots i_{\nu-1} k i_{\nu+1} \cdots i_p}.
\]

Here \( \alpha_{i_0 \cdots \tilde{i}_\nu \cdots i_p} = E_{i_\nu} \alpha_{i_0 \cdots \tilde{i}_\nu \cdots i_p} = (D\alpha)_{i_0 \cdots \tilde{i}_\nu \cdots i_p} \). In particular, Green's Theorem for a vector field \( X = X_i E_i \) can be written

\[
\int_M X_{i,d} dV = -\int_M T^i_{kl} X_k dV
\]

with \( dV = \omega^1 \wedge \cdots \wedge \omega^n \).

**Lemma 1.** For vector fields \( X \) and \( Y \) on \( M \),

\[
\int_M X_{i,j} Y_{j,i} dV = \int_M X_{i,j} Y_{j,i} dV \\
+ \int_M \left( T^k_{jk} X_{i,j} - T^k_{ik} X_{i,j} + T^k_{ij} X_{i,k} \right) Y_j dV.
\]

**Proof.** This follows from Green's Theorem and \([E_i, E_j] = -T^k_{ij} E_k\).

**Lemma 2.** For any harmonic \( p \)-form \( \alpha \), the \( L^2 \)-norm of the covariant derivative \( D\alpha \) satisfies \( \|D\alpha\|_2 \leq 10 n^4 \kappa \|\alpha\|_2 \).
Proof. We compute, using the summation convention.
\[
\|d\alpha\|_2^2 = \sum_{\mu=0}^p \int_M \alpha_{i_0 \cdots i_{\mu-1} i_{\mu}} \alpha_{i_0 \cdots i_{\mu-1} i_{\mu}} \, dV
\]
\[
+ \sum_{\mu \neq \nu} (-1)^{\mu+\nu} \int_M \alpha_{i_0 \cdots i_{\mu-1} i_{\mu}} \alpha_{i_0 \cdots i_{\nu-1} i_{\nu}} \, dV
\]
\[
+ 2 \sum_{\substack{\rho=0 \\ \rho < \nu}} ^{\nu} (-1)^{\rho+\mu+\nu} \int_M \alpha_{i_0 \cdots i_{\rho-1} i_{\rho}} \alpha_{i_0 \cdots i_{\nu-1} i_{\nu}} T_{i_\rho i_\nu}^j \alpha_{j_0 \cdots j_{\nu-1} i_{\nu}} \, dV
\]
\[
+ \sum_{\substack{\mu \neq \nu \\ \rho < \sigma}} (-1)^{\mu+\nu+\rho+\sigma} \int_M T_{i_\rho i_\sigma}^j T_{i_\sigma i_\rho}^k \alpha_{j_0 \cdots j_{\rho-1} i_{\rho}} \alpha_{k_0 \cdots k_{\sigma-1} i_{\sigma}} \, dV.
\]

The first sum is equal to \((p + 1)\|\alpha\|_2^2\). The absolute values of the third and fourth sums are bounded above by \((p + 1)n^4 \kappa \|\alpha\|_2 \|D\alpha\|_2\) and by \((p + 1)n^5 \kappa^2 \|\alpha\|_2^2\), respectively. We apply Lemma 1 and the fact that \(\delta\alpha = 0\) to obtain an upper bound \(6(p + 1)n^4 \kappa \|\alpha\|_2 \|D\alpha\|_2\) for the absolute value of the second sum: By Lemma 1,
\[
\int_M \alpha_{i_0 \cdots i_{\mu-1} i_{\mu}} \alpha_{i_0 \cdots i_{\nu-1} i_{\nu}} \, dV = \int_M \alpha_{i_0 \cdots i_{\mu-1} i_{\mu}} \alpha_{i_0 \cdots i_{\nu-1} i_{\nu}} \, dV
\]
\[
+ \int_M (T_{i_\rho k}^k \alpha_{i_0 \cdots i_{\rho-1} i_{\rho}} - T_{i_\rho k}^k \alpha_{i_0 \cdots i_{\rho-1} i_{\rho}} \alpha_{k_{i_0 \cdots i_{\rho-1} i_{\rho} i_\rho}} \, dV
\]
\[
+ T_{i_\rho i_\sigma}^j T_{i_\sigma i_\rho}^k \alpha_{k_0 \cdots k_{\rho-1} i_{\rho}} \alpha_{i_0 \cdots i_{\sigma-1} i_{\sigma}} \, dV.
\]

Since \(\delta\alpha = 0\) and \(\mu \neq \nu\), we have
\[
\alpha_{i_0 \cdots i_{\mu-1} i_{\mu}} = \pm T_{i_\rho k}^k \alpha_{k_{i_0 \cdots i_{\rho-1} i_{\rho}}} \pm \sum_{\substack{\rho=0 \\ \rho \neq \mu, \nu}} ^{\rho} \pm
\]
\[
S_{i_\rho k}^k \alpha_{k_{i_0 \cdots i_{\rho-1} i_{\rho} i_{\rho+1} \cdots i_{\rho}}} \alpha_{i_0 \cdots i_{\rho-1} i_{\rho}} \alpha_{k_{i_0 \cdots i_{\rho-1} i_{\rho} i_{\rho+1} \cdots i_{\rho}}} \, dV.
\]

After substituting these expressions, a straightforward estimate yields the required bound on the second sum. Since \(d\alpha = 0\), we obtain
\[
\|D\alpha\|_2^2 \leq 7n^4 \kappa \|\alpha\|_2 \|D\alpha\|_2 + n^5 \kappa^2 \|\alpha\|_2^2.
\]

Lemma 2 follows by completing the square.

Let \(K_\omega \subset L^2(\wedge^p T^* M)\) denote the space of parallel \(p\)-forms with respect to \(D\) and \(K_\omega^\perp\) its orthogonal complement in \(L^2\). \(K_\omega\) is the \(R\)-span of \(\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}\) and therefore has dimension \(\binom{n}{p}\).

Lemma 3. For any smooth \(p\)-form \(\alpha \in K_\omega^\perp\), \(\sqrt{\lambda_1} \|\alpha\|_2 \leq \|D\alpha\|_2\).

Proof. Recall that \(\lambda_1 = \inf \left(\|df\|_2^2/\|f\|_2^2\right)\), where the infimum is taken over all functions \(f \in C^\infty(M)\) such that \(\int_M f \, dV = 0\). We also note that \(\alpha \in K_\omega^\perp\)
if and only if each component function of \( \alpha \) is orthogonal to the constants in \( L^2 \). Thus,
\[
\|\alpha\|_2^2 = \sum_{i_1, \ldots, i_P} \|\alpha_{i_1, \ldots, i_P}\|_2^2 \leq \lambda_1^{-1} \sum_{i_1, \ldots, i_P} \|\alpha_{i_1, \ldots, i_P, j}\|_2^2 = \lambda_1^{-1} \|D\alpha\|_2^2.
\]

The hypothesis \( b_P > \binom{n}{p} \) implies that there exists a nonzero harmonic form in \( K^\perp_{\omega} \). Lemmas 2 and 3 now imply Theorem 1. To prove Theorem 2 we recall [1, Theorem 3.1]: If \( \kappa d \leq 0.05 \) then \( \lambda_1 \geq 20^{-n} d^{-2} \lambda_1^* \), where \( \lambda_1^* \) is the smallest positive eigenvalue of the Laplacean on the \( n \)-dimensional euclidean unit ball under Neumann boundary conditions. By a result of Payne and Weinberger [3], \( \lambda_1^* \geq \pi^2/4 \), and Theorem 2 follows from Theorem 1.

**References**


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