A SHORT PROOF AND
A GENERALIZATION OF MIRANDA'S EXISTENCE THEOREM

MICHAEL N. VRAHATIS

(Communicated by Paul S. Muhly)

Abstract. Miranda gave in [5] an equivalent formulation of the famous Brouwer fixed point theorem. We give a short proof of Miranda's existence theorem and then using the results obtained in this proof we give a generalization of a well-known variant of Bolzano's existence theorem. Finally, we give a generalization of Miranda's theorem.

We shall give here a short proof and a generalization of the following equivalent formulation of the famous L. E. J. Brouwer fixed point theorem [1] given by C. Miranda [5].

Theorem 1 (Miranda, 1940) [5, 10, 4, 6, 3, 11]. Let \( G = \{ x \in \mathbb{R}^n : |x_i| < L, \text{ for } 1 \leq i \leq n \} \) and suppose that the mapping \( F = (f_1, f_2, \ldots, f_n) : G \to \mathbb{R}^n \) is continuous on the closure \( \overline{G} \) of \( G \) such that \( F(x) \neq 0 = (0, 0, \ldots, 0) \) for \( x \) on the boundary \( \partial G \) of \( G \), and

(i) \( f_i(x_1, x_2, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \geq 0 \) for \( 1 \leq i \leq n \), and

(ii) \( f_i(x_1, x_2, \ldots, x_{i-1}, +L, x_{i+1}, \ldots, x_n) \leq 0 \) for \( 1 \leq i \leq n \).

Then, \( F(x) = 0 \) has a solution in \( G \).

For recent proofs of the above theorem see [10, pp. 37–38] and [3, pp. 118–119]. Theorem 1 is known to be useful in the theory of differential equations. Moreover, for some of its implementations in the case of systems of nonlinear algebraic or transcendental equations, we refer to [4, 6, 11]. Theorem 1, also, has an important property since it constitutes a straightforward generalization of the well-known and very useful, (for iterative approximate procedures for solving nonlinear equations), Bolzano's existence theorem which states: "If \( f : [a, b] \to \mathbb{R} \) is a continuous mapping and \( f(a) \) and \( f(b) \) have opposite signs, then for some \( x \in (a, b) \), it holds \( f(x) = 0 \)."
We now give a short proof of Theorem 1. For the details about degree theory used in the following proof, we refer to [7, 2, 9, 8, 3].

**Proof of Theorem 1.** Consider the homotopy,

\[ H : \bar{G} \times [0, 1] \subset \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad \text{by} \quad H(x, t) = (1 - t)F(x) + t(-x). \]

Then, \( H(x, t) \neq \theta \) for \((x, t) \in \partial G \) and \( t \in [0, 1] \). In fact, \( H(x, 0) = F(x) \neq \theta \) since \( \theta \notin F(\partial G) \), while \( H(x, 1) = -x \neq \theta \) since \( \theta \notin \partial G \); finally, \( H(x, t) = \theta \) for some \( t \in (0, 1) \) leads to the contradiction \( F(x) + t(1 - t)^{-1}(-x) = \theta \) because \( t(1 - t)^{-1} > 0 \) and by the assumptions (i) and (ii) for \( x \in \partial G \) there exist at least one \( i \) such that \( f_i(x)(-x_i) > 0 \). Thus by the homotopy invariance theorem of the degree theory, it follows that

\[ \text{deg}[F, G, \theta] = \text{deg}[H(\cdot, 0), G, \theta] = \text{deg}[H(\cdot, 1), G, \theta], \]

(where \( \text{deg}[F, G, \theta] \) indicates the topological degree of \( F \) at \( \theta \) relative to \( G \)). Hence, \( |\text{deg}[F, G, \theta]| = 1 \neq 0 \) and the result follows by the Kronecker existence theorem. \( \square \)

A corollary directly derived from the above result follows:

**Corollary 1.** Suppose that the conditions of the preceding theorem hold. Assume that \( F(x) = \theta \) has only simple solutions in \( G \), (i.e., the Jacobian determinant of \( F \) does not vanish at any solution). Then \( F(x) = \theta \) has an odd number of solutions in \( G \).

**Proof.** The result follows from the fact that \( |\text{deg}[F, G, \theta]| = 1 \), which we have determined in the proof of the previous theorem. \( \square \)

It is readily seen that Corollary 1 generalizes a well-known variant of Bolzano’s Theorem (odd number of solutions) which states: “If \( f(a) \) and \( f(b) \) have opposite signs and whenever \( f(x) = 0 \) for \( x \in (a, b) \) holds that \( f'(x) \neq 0 \), then \( f(x) = 0 \) has an odd number of solutions in \( (a, b) \).”

A generalization of Theorem 1 follows:

**Theorem 2.** Let \( \beta_1, \beta_2, \ldots, \beta_n \) be \( n \) linearly independent vectors in \( \mathbb{R}^n \), let \( \langle , \rangle \) denote the standard inner product and \( G = \{ x \in \mathbb{R}^n : |\langle \beta_i, x \rangle| < L, \text{ for } 1 \leq i \leq n \} \). Suppose that \( F = (f_1, f_2, \ldots, f_n) : \bar{G} \to \mathbb{R}^n \) is a continuous mapping such that \( F(x) \neq \theta \) for \( x \in \partial G \), and

\[ \langle F(x), \beta_i \rangle \geq 0 \quad \text{if} \quad \beta_i^\top x = -L \quad \text{for} \quad 1 \leq i \leq n, \quad \text{and} \]

\[ \langle F(x), \beta_i \rangle \leq 0 \quad \text{if} \quad \beta_i^\top x = +L \quad \text{for} \quad 1 \leq i \leq n. \]

Then, \( F(x) = \theta \) has a solution in \( G \) and, in fact, \( |\text{deg}[F, G, \theta]| = 1 \).

**Proof.** Consider the mapping,

\[ \Lambda : \mathbb{R}^n \to \mathbb{R}^n, \quad \text{by} \quad \Lambda(x) = (\langle \beta_1, x \rangle, \langle \beta_2, x \rangle, \ldots, \langle \beta_n, x \rangle). \]

Clearly, \( \Lambda \) is a one-to-one linear mapping. So,

\[ \text{deg}[F, G, \theta] = \text{deg}[\Lambda F \Lambda^{-1}, \Lambda G, \theta], \]
which reduces the present theorem to Theorem 1. Finally, following the proof of Theorem 1 we can obtain $|\deg[F, G, \theta]| = 1$. □

REFERENCES


Department of Mathematics, University of Patras, GR–261.10 Patras, Greece