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A WEAK-STAR RATIONAL APPROXIMATION PROBLEM CONNECTED WITH SUBNORMAL OPERATORS

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ABSTRACT. Let μ be a positive Borel measure on a compact subset K of the complex plane. Denote the weak-star closure in $L^{\infty}(\mu)$ of $R(K)$ by $R^{\infty}(K,\mu)$. Given $f \in R^{\infty}(K,\mu)$, denote the weak-star closure in $L^{\infty}(\mu)$ of the algebra generated by $R^{\infty}(K,\mu)$ and the complex conjugate of f by $A^{\infty}(f,\mu)$. This paper determines the structure of $A^{\infty}(f,\mu)$. As a consequence, a solution is obtained to a problem concerned with minimal normal extensions of functions of a subnormal operator.

0. INTRODUCTION

Let K be a compact subset of the complex plane $\mathbf{C}$ and let μ be a finite, positive, Borel measure on K. Define $R^{\infty}(K,\mu)$ to be the weak-star closure in $L^{\infty}(\mu)$ of $R(K)$. Here $R(K)$ denotes, as usual, the uniform closure in $C(K)$ of the holomorphic rational functions with poles off K. Given $f \in R^{\infty}(K,\mu)$, define $A^{\infty}(f,\mu)$ to be the weak-star closure in $L^{\infty}(\mu)$ of the algebra generated by $R^{\infty}(K,\mu)$ and the complex conjugate $\overline{f}$ of f. The problem addressed in this paper is that of determining when $A^{\infty}(f,\mu) = L^{\infty}(\mu)$, or more broadly, that of determining the structure of $A^{\infty}(f,\mu)$.

Although this problem is purely function-theoretic, it is equivalent to an operator-theoretic question concerning the minimal normal extension (mne) of a function of a subnormal operator. The equivalence follows. For more details, see [D].

Theorem. $A^{\infty}(f,\mu) = L^{\infty}(\mu)$ *iff* $\operatorname{mne} f(S) = f(\operatorname{mne} S)$ *for every subnormal operator S whose spectrum is contained in K and whose scalar-valued spectral measure can be taken to be μ.*

The problem was first raised in [CO] where it was solved for any polynomially convex K. In this case $R^{\infty}(K,\mu)$ reduces to $P^{\infty}(\mu)$, the weak-star closure in $L^{\infty}(\mu)$ of the holomorphic polynomials. Subsequently, the problem was

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solved for any finitely connected (and even some infinitely connected) K in [D]. Recently, in [GRT] the problem has been solved for arbitrary K by clever elementary means while in [C] a new proof of the P^∞ result of [CO] has been set forth whose novel ingredient is the use of disintegration of measures. The present paper uses the disintegration technique of [C] to recapture and improve upon the result of [GRT]. Although the resulting proof is no longer elementary, it is rather elegant.

1. Preliminaries concerning $R^\infty(K,\mu)$

For the duration of this section, K, μ, and f shall be fixed as in the introduction. The elementary theory of $R(K)$, as can be found in the second chapter of [G1], for example, shall be assumed known. It is necessary to expand the definition of $R(K)$ somewhat. Given a Borel subset Δ of K, define $R(\Delta)$ to be the uniform closure in $C(K)$ of the set of functions Φ of the form

$$\Phi(z) = \int \frac{\varphi(\zeta)}{\zeta - z}\, dA(\zeta),$$

where φ varies over all compactly supported, bounded Borel functions on $\mathbf{C}$ that vanish on Δ. Here A denotes area measure on $\mathbf{C}$. Given a complex measure σ on K, let $\hat{\sigma}(z_0) = \int \frac{d\sigma(z)}{z - z_0}$ be the Cauchy transform of σ. An elementary and useful fact which follows from Fubini's Theorem is that σ annihilates $R(\Delta)$ iff $\hat{\sigma} = 0$ A—a.e. off Δ. Thus for $\Delta = K$, the newly defined $R(\Delta)$ is indeed the old $R(K)$. Also $R(\Delta_1) \subseteq R(\Delta_2)$ whenever $\Delta_2 \subseteq \Delta_1 \subseteq K$.

The other facts needed concerning $R(\Delta)$ are not so easily seen and followed.

Proposition. (a) *$R(\Delta)$ is a uniform algebra on K* [G2, 3.2].

(b) *If $g \in R(\Delta)$ extends to be holomorphic in a neighborhood of $z_0 \in K$, then*

$$\frac{g(z) - g(z_0)}{z - z_0} \in R(\Delta)$$

[G2, 4.6 and 4.7].

(c) *Given $z_0 \in K$, the functions in $R(\Delta)$ which extend to be holomorphic near z_0 are uniformly dense in $R(\Delta)$* [G2, 4.6 and 4.8].

Define the *envelope* E of μ with respect to K to be the set of all $z \in K$ that possess a complex representing measure μ_z for $R(K)$ such that $\mu_z << \mu$ and $\mu_z(\{z\}) = 0$. For each fixed $z \in E$, the complex homomorphism of evaluation at z on $R(K)$ extends uniquely to a weak-star continuous, complex homomorphism on $R^\infty(K,\mu)$. This extension is given by the functional $g \in R^\infty(K,\mu) \mapsto \breve{g}(z) \in \mathbf{C}$ where $\breve{g}(z) = \int g\, d\mu_z$. Unfixing $z \in E$, one has a point function $\breve{g}\colon E \mapsto \mathbf{C}$ associated with each measure function $g \in R^\infty(K,\mu)$. Furthermore, $\breve{g}_n \to \breve{g}$ pointwise boundedly on E whenever $g_n \to g$ boundedly μ—a.e. in $R^\infty(K,\mu)$.

The other facts needed concerning these notions are not so easily seen and follow.

Proposition. (d) *E is a Borel set* [Ch, X.5].

(e) $R(E) \subseteq R^{\infty}(K, \mu)$ [Ch, X.6].

(f) *For any $g \in R(E)$, $\breve{g} = g|E$* [Ch, X.6].

(g) *Given $g \in R^{\infty}(K, \mu)$, there exists a sequence $\{g_n\}_{n\geq 1}$ from $R(E)$ such that $\|g_n\|_K \leq \|g\|_{\mu}$ and $g_n \to g$ pointwise μ—a.e.* [Ch, XI.10].

From all this, one can see that $\breve{g} : E \mapsto \mathbf{C}$ is a Borel function for $g \in R^{\infty}(K, \mu)$.

Finally, given a Borel subset Δ of K and a finite, positive, Borel measure σ on K, define $R^{\infty}(\Delta, \sigma)$ to be the weak-star closure in $L^{\infty}(\sigma)$ of $R(\Delta)$. The characterization of annihilating measures mentioned earlier yields a Hartogs-Rosenthal Theorem for $R(\Delta)$, to wit, $R(\Delta) = C(K)$ whenever $A(\Delta) = 0$. The following is then immediate.

Proposition. (h) *$R^{\infty}(\Delta, \sigma) = L^{\infty}(\sigma)$ whenever $A(\Delta) = 0$.*

2. Preliminaries concerning the disintegration of measures

Let X be a locally compact, separable, metric space and let μ be a finite, positive, Borel measure on X. As opposed to $L^{\infty}(\mu)$ and $L^1(\mu)$, which are Banach spaces of equivalence classes of μ-measurable functions on X, it will also be necessary to consider $\mathscr{L}^{\infty}(\mu)$ and $\mathscr{L}^1(\mu)$, which are the corresponding pseudo-normed spaces of Borel functions on X. Fix $\varphi \in \mathscr{L}^{\infty}(\mu)$. Denote the μ-essential range of φ by Y and consider the finite, positive, Borel measure ν on Y defined by $\nu(\Delta) = \mu(\varphi^{-1}(\Delta))$. Designate the set of Borel probability measures on X by $\mathscr{P}(X)$.

An assignment $\zeta \in Y \mapsto \lambda_{\zeta} \in \mathscr{P}(X)$ is called a *disintegration* of μ with respect to φ iff for every $\psi \in \mathscr{L}^1(\mu)$ one has

(1) $\psi \in \mathscr{L}^1(\lambda_{\zeta})$ for ν—a.e. $\zeta \in Y$,

(2) the function $\zeta \in Y \mapsto \int \psi(z)\, d\lambda_{\zeta}(z) \in \mathbf{C}$ determines an element of $L^1(\nu)$, and

(3) $\int \psi(z)\, d\mu(z) = \int \left\{ \int \psi(z)\, d\lambda_{\zeta}(z) \right\} d\nu(\zeta)$.

A clean and elegant proof of the following result is set forth in the second section of [AK]. Note where the measures of any disintegration must be concentrated!

Theorem. *Suppose X, μ, and φ are as mentioned. Then there exists a disintegration $\zeta \in Y \mapsto \lambda_{\zeta} \in \mathscr{P}(X)$ of μ with respect to φ such that λ_{ζ} is concentrated on $\varphi^{-1}(\zeta)$ for ν—a.e. $\zeta \in Y$. Also, if $\zeta \in Y \mapsto \lambda'_{\zeta} \in \mathscr{P}(X)$ is another disintegration of μ with respect to φ, then $\lambda'_{\zeta} = \lambda_{\zeta}$ for ν—a.e. $\zeta \in Y$.*

One can easily see the following three facts from the definition of the measure ν and the definition of the disintegration $\zeta \in Y \mapsto \lambda_{\zeta} \in \mathscr{P}(X)$.

Proposition. (i) *If* $\mu(\varphi^{-1}(\zeta)) = 0$, *then* $\nu(\{\zeta\}) = 0$.

(j) *If* $\mu(\varphi^{-1}(\zeta)) > 0$, *then* $\nu(\{\zeta\}) > 0$.

(k) *If* $\mu(\varphi^{-1}(\zeta)) > 0$, *then* $\lambda_\zeta = [1/\mu(\varphi^{-1}(\zeta))]\cdot\mu|\varphi^{-1}(\zeta)$.

3. The main result

For the duration of this section, K, μ, and f shall be fixed as in the introduction. Arbitrarily select a Borel representative from the equivalence class of $f \in L^\infty(\mu)$ and designate it too by f. Thus, f is being viewed as an element of $\mathscr{L}^\infty(\mu)$. Taking the X, μ, and φ of the previous section to be the K, μ, and f of this section, get a disintegration $\zeta \in Y \mapsto \lambda_\zeta \in \mathscr{P}(K)$ of μ with respect to f.

Suppose g' and $g'' \in \mathscr{L}^\infty(\mu)$ represent the same element of $L^\infty(\mu)$. Then $g' = g''$ μ—a.e. and so by setting $\psi = |g' - g''|$ in (3) of the last section, one sees that $g' = g''$ λ_ζ—a.e. for ν—a.e. $\zeta \in Y$, i.e., g' and $g'' \in \mathscr{L}^\infty(\lambda_\zeta)$ represent the same element of $L^\infty(\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$. Because of this, one may properly construe the phrase "$g \in R^\infty(E,\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$" of the lemma that follows to mean that g' represents an element of $R^\infty(E,\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$ where $g' \in \mathscr{L}^\infty(\mu)$ is any arbitrarily chosen representative of $g \in L^\infty(\mu)$. In what follows, for the sake of notational simplicity and because no harm thereby results, such careful distinctions between elements of $L^\infty(\mu)$ and their corresponding representatives in $\mathscr{L}^\infty(\mu)$ shall be left tacit.

Lemma. $A^\infty(f,\mu) = \{g \in L^\infty(\mu) : g \in R^\infty(E,\lambda_\zeta)$ *for* ν—*a.e.* $\zeta \in Y\}$.

Proof. Denote the right-hand side of the lemma's equality by $\mathscr{A}^\infty(f,\mu)$. Since $R(K) \subseteq R(E)$, $R(K) \subseteq \mathscr{A}^\infty(f,\mu)$. Since λ_ζ is concentrated on $f^{-1}(\zeta)$ for ν—a.e. $\zeta \in Y$, f is constant as an element of $L^\infty(\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$ and so $\bar{f} \in \mathscr{A}^\infty(f,\mu)$. Clearly $\mathscr{A}^\infty(f,\mu)$, being an algebra, must then contain the algebra generated by $R(K)$ and $\bar{f}$. But the weak-star closure in $L^\infty(\mu)$ of the algebra generated by $R(K)$ and $\bar{f}$ is just $A^\infty(f,\mu)$. Hence to get $A^\infty(f,\mu) \subseteq \mathscr{A}^\infty(f,\mu)$, it suffices to show $\mathscr{A}^\infty(f,\mu)$ weak-star closed in $L^\infty(\mu)$. By the Krein-Schmulian Theorem, it in turn suffices to show ball $\mathscr{A}^\infty(f,\mu)$ weak-star closed in $L^\infty(\mu)$.

To this end, let g be in the weak-star closure in $L^\infty(\mu)$ of ball $\mathscr{A}^\infty(f,\mu)$. Since $L^2(\mu) \subseteq L^1(\mu)$, g is in the weak closure in $L^2(\mu)$ of ball $\mathscr{A}^\infty(f,\mu)$. But the weak and strong closures of a convex subset of a Banach space coincide, so g is in the strong closure in $L^2(\mu)$ of ball $\mathscr{A}^\infty(f,\mu)$. Accordingly, take $\{g_n\}_{n\geq 1}$ from ball $\mathscr{A}^\infty(f,\mu)$ such that $g_n \to g$ in $L^2(\mu)$. Passing to a subsequence, one may also assume that $g_n \to g$ μ—a.e.

Now select a Borel subset Δ of K so that

(i) μ is concentrated on Δ,

(ii) $|g_n| \leq 1$ on Δ for $n \geq 1$, and

(iii) $g_n \to g$ pointwise on Δ.

Setting $\psi = \chi_{K\setminus\Delta}$ in (3) of the last section and using (i), one sees that

(iv) λ_ζ is concentrated on Δ for ν—a.e. $\zeta \in Y$.

Because of (ii), (iii), and (iv), the Lebesgue Dominated Convergence Theorem implies that $g_n \to g$ weak-star in $L^\infty(\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$. Since $\{g_n\}_{n\geq 1} \subseteq \mathscr{A}^\infty(f,\mu)$, $\{g_n\}_{n\geq 1} \subseteq R^\infty(E,\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$. Clearly then, $g \in R^\infty(E,\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$ and so $g \in \text{ball}\,\mathscr{A}^\infty(f,\mu)$. Thus, ball $\mathscr{A}^\infty(f,\mu)$ is weak-star closed in $L^\infty(\mu)$.

Having just established the inclusion $A^\infty(f,\mu) \subseteq \mathscr{A}^\infty(f,\mu)$, to obtain equality it suffices, by the Hahn-Banach Theorem, to show that $\mathscr{A}^\infty(f,\mu)$ is annihilated by any weak-star continuous annihilator of $A^\infty(f,\mu)$.

To this end, let $h \in L^1(\mu)$ annihilate $A^\infty(f,\mu)$. Choose $\{g_n\}_{n\geq 1}$ uniformly dense in $R(E)$. Suppose $p(\zeta,\bar\zeta)$ is a polynomial in ζ and $\bar\zeta$. By Proposition (e), $p(f,\bar f)g_n \in A^\infty(f,\mu)$. Hence setting $\psi = p(f,\bar f)g_n h$ in (3) of the last section, one has

$$\begin{aligned} 0 &= \int p(f,\bar f) g_n h\, d\mu \\ &= \int \left\{ \int p\big(f(z), \overline{f(z)}\big) g_n(z) h(z)\, d\lambda_\zeta(z) \right\} d\nu(\zeta) \\ &= \int p(\zeta,\bar\zeta) \left\{ \int g_n(z) h(z)\, d\lambda_\zeta(z) \right\} d\nu(\zeta). \end{aligned}$$

Here the last equality follows since λ_ζ is concentrated on $f^{-1}(\zeta)$ for ν—a.e. $\zeta \in Y$, thus making $p\big(f(z),\overline{f(z)}\big) = p(\zeta,\bar\zeta)$ for λ_ζ—a.e. z for ν—a.e. $\zeta \in Y$. By the Stone-Weierstrass Theorem, the $L^1(\nu)$ function $\zeta \in Y \mapsto \int g_n h\, d\lambda_\zeta \in \mathbf{C}$ annihilates all continuous functions on Y. Hence $\int g_n h\, d\lambda_\zeta = 0$ for ν—a.e. $\zeta \in Y$. But $n \geq 1$ is arbitrary and $\{g_n\}_{n\geq 1}$ is weak-star dense in $R^\infty(E,\lambda_\zeta)$ for every $\zeta \in Y$, so it follows that $h \in L^1(\lambda_\zeta)$ annihilates $R^\infty(E,\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$.

Now take $g \in \mathscr{A}^\infty(f,\mu)$. Then, by definition, $g \in R^\infty(E,\lambda_\zeta)$ for ν—a.e. $\zeta \in Y$. Thus, setting $\psi = gh$ in (3) of the last section, one sees that

$$\int gh\, d\mu = \int \left\{ \int g(z) h(z)\, d\lambda_\zeta(z) \right\} d\nu(\zeta) = 0.$$

Thus, $h \in L^1(\mu)$ annihilates $\mathscr{A}^\infty(f,\mu)$ and so $\mathscr{A}^\infty(f,\mu) = A^\infty(f,\mu)$. □

Lemma. $R^\infty(E,\lambda_\zeta) = R^\infty(\check{f}^{-1}(\zeta),\, \lambda_\zeta)$ *for* ν*—a.e.* $\zeta \in Y$.

Proof. By Proposition (g), there exists a sequence $\{f_n\}_{n\geq 1}$ from $R(E)$ such that

(i) $$f_n \to f \qquad \text{boundedly } \mu\text{—a.e.}$$

Let Δ be a Borel subset of K on which μ is concentrated such that $f_n \to f$ pointwise boundedly on Δ. Setting $\psi = \chi_{K\setminus\Delta}$ in (3) of the last section, one sees that λ_ζ is concentrated on Δ for ν—a.e. $\zeta \in Y$. Thus $f_n \to f$ boundedly λ_ζ—a.e. for ν—a.e. $\zeta \in Y$. Since λ_ζ is concentrated on $f^{-1}(\zeta)$ for ν—a.e. $\zeta \in Y$; it follows that for ν—a.e. $\zeta \in Y$,

(ii) $$f_n \to \zeta \qquad \text{boundedly } \lambda_\zeta\text{—a.e.}$$

Consequently, it suffices to show $R^\infty(E,\lambda_\zeta) = R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta)$ for any $\zeta \in Y$ such that (ii) holds.

Since $\check{f}^{-1}(\zeta) \subseteq E$, $R^\infty(E,\lambda_\zeta) \subseteq R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta)$. Thus, to obtain equality it suffices, by the Hahn-Banach Theorem, to show that $R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta)$ is annihilated by any weak-star continuous annihilator of $R^\infty(E,\lambda_\zeta)$.

To this end, let $h \in L^1(\lambda_\zeta)$ annihilate $R^\infty(E,\lambda_\zeta)$ and consider the measure σ defined by $d\sigma = h\,d\lambda_\zeta$. By hypothesis, σ annihilates $R(E)$, i.e., $\hat{\sigma} = 0$ A—a.e. off E. Since $\sigma << \lambda_\zeta$, to prove that h annihilates $R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta)$ it suffices to show that σ annihilates $R(\check{f}^{-1}(\zeta))$, i.e., $\hat{\sigma} = 0$ A—a.e. off $\check{f}^{-1}(\zeta)$. Let $\tilde{\sigma}(z_0) = \int \frac{d|\sigma|(z)}{|z-z_0|}$ be the Newtonian potential of σ. Since $\tilde{\sigma} < \infty$ A—a.e. on $\mathbf{C}$, it in turn suffices to show $\hat{\sigma}(z_0) = 0$ whenever

(iii) $$z_0 \in E \setminus \check{f}^{-1}(\zeta) \qquad \text{and} \qquad \tilde{\sigma}(z_0) < \infty.$$

By Proposition (c), there exists a sequence $\{g_n\}_{n\geq 1}$ from $R(E)$, with each g_n extending to be holomorphic near z_0, such that

(iv) $$\|g_n - f_n\|_K \to 0.$$

By (i) and (iv), $g_n \to f$ boundedly μ—a.e. and so $\check{g}_n \to \check{f}$ pointwise boundedly on E. But then by (iii) and Proposition (f),

(v) $$g_n(z_0) = \check{g}_n(z_0) \to \check{f}(z_0).$$

Since $\sigma << \lambda_\zeta$, by (ii) and (iv) one has

(vi) $$g_n \to \zeta \qquad \text{boundedly } \sigma\text{—a.e.}$$

Because of (iii), (v), and (vi), the Lebesgue Dominated Convergence Theorem implies that

(vii) $$\lim_{n\to\infty} \int \frac{g_n(z) - g_n(z_0)}{z - z_0}\, d\sigma(z) = \int \frac{\zeta - \check{f}(z_0)}{z - z_0}\, d\sigma(z).$$

The left-hand side of (vii) is zero since σ annihilates $R(E)$ and each $(g_n(z) - g_n(z_0))/(z - z_0) \in R(E)$ by Proposition (b). Since $\zeta - \check{f}(z_0)$ is constant as far as the integration is concerned, the right-hand side of (vii) is just $\{\zeta - \check{f}(z_0)\}\hat{\sigma}(z_0)$. Thus, $\{\zeta - \check{f}(z_0)\}\hat{\sigma}(z_0) = 0$. By (iii), $\zeta - \check{f}(z_0) \neq 0$, so it must be that $\hat{\sigma}(z_0) = 0$. □

Let $\tilde{a}_1, \tilde{a}_2, \cdots$ be an enumeration of the at most countable set of points $\zeta \in \mathbf{C}$ such that $\mu(f^{-1}(\zeta)) > 0$. Set $\tilde{\mu}_i = \mu | f^{-1}(\tilde{a}_i)$ and $\tilde{\mu}_s = \mu - \sum \tilde{\mu}_i$. The main result of [GRT] states that $A^\infty(f,\mu) = L^\infty(\mu_s) \oplus \sum \oplus R^\infty(K,\tilde{\mu}_i)$ where the right-hand side of the equality is interpreted to mean $\{g \in L^\infty(\mu) : g \in R^\infty(K,\tilde{\mu}_i)$ for each $i = 1,2,\ldots\}$. Compare this to the main result of the present paper, which follows immediately.

Theorem. *Let $a_1, a_2, \ldots$ be an enumeration of the at most countable set of points $\zeta \in \mathbf{C}$ such that both $\mu(f^{-1}(\zeta)) > 0$ and $A(\check{f}^{-1}(\zeta)) > 0$. Set $\mu_i = \mu|f^{-1}(a_i)$ and $\mu_s = \mu - \sum \mu_i$. Then*

$$A^\infty(f,\mu) = L^\infty(\mu_s) \oplus \sum \oplus R^\infty(\check{f}^{-1}(a_i),\mu_i).$$

Proof. Set $\Omega = \{\zeta \in \mathbf{C} : \mu(f^{-1}(\zeta)) > 0$ and $A(\check{f}^{-1}(\zeta)) > 0\}$ and $\Omega' = \{\zeta \in \mathbf{C} : \mu(f^{-1}(\zeta)) = 0$ and $A(\check{f}^{-1}(\zeta)) > 0\}$. Note that both Ω and Ω' are countable with $\Omega = \{a_1, a_2, \ldots\}$ and $\Omega \cup \Omega' = \{\zeta \in \mathbf{C} : A(\check{f}^{-1}(\zeta)) > 0\}$. Then

$$\begin{aligned}
A^\infty(f,\mu) &= \{g \in L^\infty(\mu) : g \in R^\infty(E,\lambda_\zeta) \text{ for } \nu\text{—a.e. } \zeta \in Y\} \\
&\hspace{18em}\text{(by Lemma One)} \\
&= \{g \in L^\infty(\mu) : g \in R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta) \text{ for } \nu\text{—a.e. } \zeta \in Y\} \\
&\hspace{18em}\text{(by Lemma Two)} \\
&= \{g \in L^\infty(\mu) : g \in R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta) \text{ for } \nu\text{—a.e. } \zeta \in \Omega \cup \Omega'\} \\
&\hspace{18em}\text{(by Proposition (h))} \\
&= \{g \in L^\infty(\mu) : g \in R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta) \text{ for } \nu\text{—a.e. } \zeta \in \Omega\} \\
&\hspace{18em}\text{(by Proposition (i))} \\
&= \{g \in L^\infty(\mu) : g \in R^\infty(\check{f}^{-1}(\zeta),\lambda_\zeta) \text{ for every } \zeta \in \Omega\} \\
&\hspace{18em}\text{(by Proposition (j))} \\
&= \{g \in L^\infty(\mu) : g \in R^\infty(\check{f}^{-1}(\zeta),\mu|f^{-1}(\zeta)) \text{ for every } \zeta \in \Omega\} \\
&\hspace{18em}\text{(by Proposition (k))} \\
&= \{g \in L^\infty(\mu) : g \in R^\infty(\check{f}^{-1}(a_i),\mu_i) \text{ for each } i = 1,2,\ldots\} \\
&= L^\infty(\mu_s) \oplus \sum \oplus R^\infty(\check{f}^{-1}(a_i),\mu_i). \qquad \square
\end{aligned}$$

Of course there may be no a_i's, in which case one has the following:

Corollary. $A^\infty(f,\mu) = L^\infty(\mu)$ *whenever each $\zeta \in \mathbf{C}$ is such that either*

$$\mu(f^{-1}(\zeta)) = 0 \quad \textit{or} \quad A(\check{f}^{-1}(\zeta)) = 0.$$

A necessary and sufficient condition for weak-star density is contained in the following:

Corollary.

$$A^\infty(f,\mu) = L^\infty(\mu) \quad \textit{iff}\ R^\infty(\check{f}^{-1}(\zeta),\mu|f^{-1}(\zeta)) = L^\infty(\mu|f^{-1}(\zeta))$$

for each $\zeta \in \mathbf{C}$.

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REFERENCES

[AK] M. B. Abrahamse and T. L. Kriete, *The spectral multiplicity of a multiplication operator*, Indiana Univ. Math. J. **22** (1973), 845–857.

[Ch] J. Chaumat, *Adherence faible étoile d'algebres de fractions rationelles*, Publications Mathématique d'Orsay, No. 147.

[C] J. B. Conway, *'The minimal normal extension of a function of a subnormal operator' in Proceedings of the Special Year in Operator Theory at the University of Illinois*, Cambridge Univ. Press, Cambridge (to appear).

[CO] J. B. Conway and R. F. Olin, *A functional calculus for subnormal operators*, II, Memoirs Amer. Math. Soc. No. 184, Providence, R.I.

[D] J. J. Dudziak, *The minimal normal extension problem for subnormal operators*, J. Funct. Anal. **65** (1986), 314–338.

[G1] T. W. Gamelin, *Uniform algebras*, Prentice Hall, Englewood Cliffs, N.J.

[G2] ——, *Rational approximation theory*, Univ. California at Los Angeles Lecture Notes (unpublished), Los Angeles, CA.

[GRT] T. W. Gamelin, P. Russo and J. E. Thomson, *A Stone-Weierstrass theorem for weak-star approximation by rational functions*, J. Funct. Anal. (to appear).

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