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ON EXTENDING ACTIONS

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ABSTRACT. Consider a Polish topological group G acting via J on a substandard (= countably generated) Borel space. **Theorem 1.** *Any such "Borel action" can be extended to a Borel action* $J': G \times X' \to X'$ *where* X' *is coanalytic.* (Theorem 3 gives an analogue for continuous actions.) **Corollary 2.** *The result "in any Borel action, orbits are Borel" implies the (well-known) result "all such orbits are absolutely Borel".*

A *Borel space* is a set Y together with a σ-field of subsets of Y. A Borel space Y is called *standard* [resp. *substandard*] if it is isomorphic to the Borel structure of some Polish [resp. separable metrizable] topological space. If Y is standard, X is called an *analytic subset of* Y if for some standard Z and some B Borel in $Y \times Z$, $X = \{y \in Y: (\exists z \in Z)(y, z) \in B\}$. X is a *coanalytic* subset of Y if the same holds with '$\exists$' replaced by '$\forall$'. X is called plain *analytic* [*coanalytic*] if it is an analytic [resp. coanalytic] subset of some standard space.

Suppose G is a Polish topological group which acts on a substandard Borel space X via a Borel map $J: G \times X \to X$. (Here "acts" means that for any $g, h \in G$ and any $x \in X$, $ex = x$ and $g(hx) = (gh)x$ where $gx = J(g, x)$.) Then (J, G, X) is called a *Borel action.*

We will establish in Theorem 1 that *any Borel action* (J, G, X) *can be extended to a Borel action* (J', G, X') *where* X' *is coanalytic.*

Theorem 1 is applied in Corollary 2, which was its original motivation. In any Borel action (J, G, X), $\mathscr{O} \subseteq X$ is called an *orbit* if for some $x_0 \in X$, $\mathscr{O} = \{gx_0: g \in G\}$. Douglas Miller [3] proved that in any Borel action (J, G, X), all orbits are Borel (in X). In so doing, he improved and extended various overlapping results of C. Kuratowski ([1, p. 377]), G. Mackey [1957], J. Dixmier [1962], D. Scott [1964], C. Ryll-Nardzewski [1964], and this author [4]. (Missing references can be found in Miller [3].) In fact, Miller gave two proofs of his result. In one he used (among other things) cross-sections or selectors, as had Mackey, Dixmier, and Ryll-Nardzewski. In the other proof Miller extended to Borel spaces the "$*$-transform" method of the author [4] (which is loosely related to Scott's proof). Now, in fact, all of these authors

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using cross-sections showed that all these orbits $\mathscr{O}$ are *absolutely Borel* (i.e., $\mathscr{O}$ is a Borel set in some standard space). On the other hand, the second proof only shows that each orbit in (J, G, X) is a Borel set in X.

This situation is "resolved" in Corollary 2: *The result that in every Borel action orbits are Borel implies that all such orbits are* absolutely *Borel.*

(J, G, X) is called a *continuous action* if G, as before, acts on a separable metrizable *topological space* X via the *continuous* map $J: G \times X \to X$. We prove an analogue of Theorem 1 in Theorem 3: *Any continuous action* (J, G, X) *can be extended to a continuous action* (J', G, X'), *where* X' *is coanalytic* (i.e., *the Borel structure of* X' *is coanalytic*).

0. Known facts

A Borel space X is Borel (resp. coanalytic) in some standard Y if and only if X is Borel (resp. coanalytic) in every standard $Z \supseteq X$ (cf. [2, §38, VII]).

Here are two well-known facts which can be found in stronger forms in [2, §35, VI], but are more easily just proved here.

(1) *Every substandard* X *is isomorphic to a subspace of* 2^ω.

Indeed, let $U_1, \dots, U_n, \dots$ be a set of σ-Boolean generators for the family of Borel sets of X. If $x \in X$, put $F(x)(n) = 1$ if $x \in U_n$, $F(x) = 0$ otherwise. One can now easily check that F is a Borel isomorphism of X onto $F[X]$, proving (1). (It follows that 'substandard' = 'subspace of standard'.)

(2) *Suppose* X_2 *is substandard,* $X_1 \subseteq X_2$, *and* $F_1: X_1 \to 2^\omega$ *is a Borel map. Then* F_1 *can be extended to a Borel map* $F_2: X_2 \to 2^\omega$.

To see this put $U_n = \{x \in X_1 : F_1(x)(n) = 1\}$ for each n. Then $F: X_1 \to 2^\omega$ and $U_1, \dots, U_n, \dots$ are as in the proof of (1). For each n, choose U_n' Borel in X_2 such that $U_n' \cap X_1 = U_n$. Define $F': X^\omega \to 2$ as F was defined in the proof of (1) but now with respect to X_2 and $U_0', \dots, U_n', \dots$. It is easy to see F' is as desired.

The reader is referred to [2, §35, I, Theorem 1], for the proof of

(2′) *Suppose* X_2 *is a separable, metrizable, topological space;* $X_1 \subseteq X_2$, Y *is Polish, and* $F: X_1 \to Y$ *is continuous. Then* F *can be extended to a continuous map* $F': W \to Y$, *where* W *is a* G_δ *subset of* X_2.

1. Extending Borel actions

Theorem 1. *Any Borel action* $J: G \times X \to X$ *can be extended to a Borel action* $J': G \times X' \to X'$, *where* X' *is coanalytic.*

Proof. By (1), X is a subspace of an isomorph Y of 2^ω. Thus $J: G\times X \to Y$ and $G\times X \subseteq G\times Y$. By (2), J can be extended to a Borel map $J^\frown: G\times Y \to Y$.

($J^\frown$ need not be an action; however, we still write $gy = J^\frown(g, y)$.) Put

$$X' = \{y \in Y : ey = y \wedge (\forall g, h \in G)(g(hy) = (gh)y)\}.$$

Clearly, $X \subseteq X'$. Since Y and G are standard while J' and group multiplication are both Borel maps, it is easy to check that X' is coanalytic in Y.

Suppose $x' \in X'$ and $k \in G$. We claim that $kx' \in G'$. Indeed, since $x' \in X'$, $e(kx') = (ek)x' = kx'$. Also, for any g, $h \in G$, since $x' \in X'$, we have $g(h(kx')) = g((hk)x') = (g(hk))x' = ((gh)k)x' = (gh)(kx')$. Thus $kx' \in X'$, as claimed.

Now let $J' = J^\frown \restriction (G \times X')$. Since $X \subseteq X'$, J' extends J. As we just proved, $J' : G \times X' \to X'$, Clearly J' is a Borel map. The conditions $ex' = x'$ and $g(hx') = (gh)x'$ (for all $x' \in X'$) were built into the definition of X'. Thus (J', G, X') is a Borel action extending J with X' coanalytic, and Theorem 1 is proved.

2. Application to orbits

Corollary 2. *The result that in every Borel action, all orbits are Borel, implies that all such orbits are absolutely Borel.*

Proof. Assume that in all Borel actions, orbits are Borel. Let $\mathcal{O}$ be an orbit in a Borel action (J, G, X). Then $(G, J \restriction (G \times \mathcal{O}), \mathcal{O})$ is also a Borel action. Thus (as was long known) it is enough to consider only the case $\mathcal{O} = X$.

By Theorem 1 we can extend J to a Borel action $J' : G \times X' \to X'$ where X' is a coanalytic subset of some standard Y. As was also long known, $\mathcal{O} = X$ is an analytic subset of Y. (Indeed, if $x_0 \in \mathcal{O}$ then $\mathcal{O} = \{x \in Y : (\exists g \in G)(x = J(g, x_0))\}$.) It follows easily that $\mathcal{O}$ is an analytic subset of Y. On the other hand, $\mathcal{O}$ is clearly an orbit in the action (J', G, X'), so $\mathcal{O}$ is Borel in X', by our general assumption. Since, in turn, X' is coanalytic in Y, it follows very easily that $\mathcal{O}$ is coanalytic in Y. Thus $\mathcal{O}$ is both analytic and coanalytic in Y; so by Souslin's Theorem ([2, §39, III, Cor. 1]), $\mathcal{O}$ is Borel in Y. Hence, since Y is standard, $\mathcal{O}$ is absolutely Borel, as was to be proved.

3. Extending continuous actions

Theorem 3. *Any continuous action (J, G, X) can be extended to a continuous action (J', G, X') where X' is coanalytic.*

Proof. The separable metrizable space X is a subspace of some Polish space Y (using the "completion"). Thus $J : G \times X \to Y$ is continuous and $G \times X \subseteq G \times Y$. Hence, by $(2')$, J can be extended to a continuous map $J^\frown : W \to Y$ where W is a G_δ subset of the Polish space $G \times Y$. It follows that W is Polish (cf. [2, §33, VI]). Write gy for $J^\frown(g, y)$ (which may not exist). Put

$$X' = \{y \in Y : \text{(A) } (\forall h \in G)\ (hy \text{ exists, i.e., } (h\ y) \in W) \wedge \text{(B) } ey = y \wedge \wedge \text{(C) } (\forall g, h \in G)\ (g(hy) \text{ exists and } = (gh)y)\}.$$

Clearly $X \subseteq X'$. It is easy to check that X' is coanalytic in Y— hence coanalytic. (As an example of the arguments needed, we show that $Y_1 = \{y \in Y : ey$ exists and $= y\}$ is Borel in Y. Clearly $Y_2 = \{y \in Y : (e, y) \in W\}$ is Borel in Y. Also $y \in Y_2 \mapsto J^\frown(e, y)$ is a Borel map of Y_2 to Y. Hence $Y_1 = \{y \in Y_2 : J^\frown(e, y) = y\}$ is Borel in Y_2, and hence in Y.) Also, by (A), $G \times X' \subseteq W$. Put $J' = J^\frown \restriction (G \times X')$. Clearly $J' : G \times X' \to Y$ and J' is continuous.

Let $x' \in X'$ and $k \in G$. We claim that $kx' \in X'$ (kx' exists by (A)). First we show (A) holds (for $y = kx'$). Let $h \in G$. By (C) (for $y = x'$), $h(kx')$ exists, as desired. Next we want to see that (B) holds (for $y = kx'$). But by (C) (for $y = x'$), $e(kx')$ exists and $= (ek)x' = kx'$, as desired. Finally, we show (C) holds (for $y = kx$). Let g, $h \in G$. By (C) (for $y = x'$), $g((hk)x')$ exists and $= (g(hk))x' = ((gh)k)x' =$ (by (C)) $(gh)(kx')$. Also by (C), $h(kx')$ exists and $= (hk)x'$; hence $g(h(kx'))$ exists and $= g((hk)x')$, which $= (gh)(kx')$ by the previous sentence. Thus (C) holds for $y = kx'$.

It only remains to show that J' is an action. But this has been built into the definition of X'. Thus (J', G, X') is a continuous action extending J, with X' coanalytic, and Theorem 3 is proved.

Remark. In Theorems 1 and 3 (but not Corollary 2), the hypothesis that G is a Polish topological group can be replaced by the assumption that G is an analytic Borel group. The proofs above only need obvious changes.

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