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INNER GRADINGS AND GALOIS EXTENSIONS WITH NORMAL BASIS

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ABSTRACT. We prove that for G a finite group, a G-graded Azumaya algebra over a commutative ring has inner grading if and only if an associated Galois extension has normal basis.

INTRODUCTION

In this note, we prove the following Skolem-Noether type theorem for graded Azumaya algebras over a commutative ring k. If G is a finite group and A a G-graded Azumaya algebra over k, then the G-grading on A is inner if and only if the Galois extension $(A\#(kG)^*)^A$ has normal basis. In particular, for k semilocal or von Neumann regular, every G-grading of a k-Azumaya algebra is inner. This result extends the work of Osterburg and Quinn [8] who proved a similar result for k a field and A strongly G-graded.

Our main theorem parallels the well-known situation for A an Azumaya kG-module algebra. For an automorphism Ω of A is inner if and only if $A(\Omega) = \{x \in A\colon ax = x\Omega(a) \text{ for all } a \in A\}$ is a free k-module of rank 1, and thus kG acts innerly on A if and only if the Galois kG-object $\bigoplus_{\sigma\in G} A(\sigma)$ has normal basis.

1. PRELIMINARIES

Throughout, k will denote a (trivially graded) commutative ring with 1 and G a finite group. Unless otherwise specified, $\otimes$, Hom, etc. should be understood to be over k, algebras and modules are k-algebras and k-modules, etc.

The Hopf algebra $(kG)^*$ is the dual of the group ring kG; $(kG)^*$ has a free basis over k, namely the projections p_σ, $\sigma \in G$, where $p_\sigma(\tau) = \delta_{\sigma,\tau}$. Thus the p_σ are a set of orthogonal idempotents with $\sum_{\sigma\in G} p_\sigma = 1$.

A k-algebra A is called a G-graded algebra if $A = \bigoplus_{\sigma\in G} A_\sigma$, $A_\sigma A_\tau \subseteq A_{\sigma\tau}$ and $k \subseteq A_1$. A is a G-graded algebra if and only if A is a $(kG)^*$-module

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algebra. Thus we may form the smash product $A\#(kG)^*$ where $A\#(kG)^* = A \otimes (kG)^*$ as a k-module and has multiplication defined by $(a\#p_\sigma)(b\#p_\tau) = ab_{\sigma\tau^{-1}}\#p_\tau$, b_γ being the γ-th homogeneous component of $b \in A$. G acts as a group of automorphisms on $A\#(kG)^*$ by $\sigma(a\#p_\tau) = a\#p_{\tau\sigma^{-1}}$; $(A\#(kG)^*)^G = A$. (See [5] for further details of this construction.)

A $(kG)^*$-module algebra A is said to have inner grading if there are maps u and v from G to A such that for all $a \in A$, $\gamma \in G$,

$$a_\gamma = \sum_{\alpha \in G} u(\gamma\alpha)av(\alpha^{-1}).$$

Then, since $1 \in A_1$,

$$\sum_{\alpha \in G} u(\gamma\alpha)v(\alpha^{-1}) = \delta_{\gamma,1}. \tag{1}$$

It was noted in [8] that for A finite dimensional over a field k, we also have

$$\sum_{\alpha \in G} v(\gamma\alpha)u(\alpha^{-1}) = \delta_{\gamma,1}. \tag{2}$$

The equation (2) holds also for A finitely generated projective over k, k a commutative ring. Equation (1) says that u is right invertible in the convolution algebra $C = \mathrm{Hom}((kG)^*, A)$ (see [10, Chapter IV] for the definition of C). But then m_u, multiplication on the right by u, is a k-module isomorphism from C to Cu, so that, at each localization of k, C and Cu are free of the same rank. Thus, the identity map from Cu to C is an isomorphism at each localization, and thus is an isomorphism. Then $C = Cu$ and u is left invertible in C. Since left and right inverses are equal, (2) holds.

Recall that a (not necessarily commutative) k-algebra S is called a Galois extension of k with group G or a Galois $(kG)^*$-object [3] if G acts as a group of automorphisms on S such that $S^G = k$ and the map $\Gamma: S \otimes S \to S \otimes (kG)^*$ defined by $\Gamma(s \otimes t) = \sum_{\sigma \in G} s\sigma(t) \otimes p_\sigma$ is an SG-module isomorphism where the G-action on $S \otimes S$ is induced by the G-action on the second factor and the G-action on $S \otimes (kG)^*$ is induced by the usual G-action on $(kG)^*$.

A Galois extension S is said to have normal basis if S is isomorphic to $(kG)^*$ as $(kG)^*$-comodules (or, equivalently, as kG-modules). S has normal basis if and only if there exists $x \in S$ such that $\{\sigma(x) : \sigma \in G\}$ is a free basis for the k-module S.

Note that the definition of an inner grading comes from Sweedler's more general definition of an inner Hopf algebra action [9]. This more general setting for Skolem-Noether type results is studied in [2] for k a field; if the Hopf algebra H is $(kG)^*$, the results of [8] are recovered. The method of proof in [2, 8] and in this paper are closely related. In fact, using [4, 3.4], one easily sees that for A strongly G-graded, the Galois extension E of [8] and the Galois extension $S(A)$ described below are isomorphic Galois extensions of k with group G.

2. The main theorem

From now on, A will denote a finitely generated projective central separable G-graded algebra. We define a Galois extension $S(A)$, associated with A, as follows. The algebra $A\#(kG)^*$ has a left $A \otimes A^o$-module structure defined by $a \otimes b^o \circ x = a\#1 \circ x \circ b\#1$. We form $S(A) = (A\#(kG)^*)^A = \{x \in A\#(kG)^* : a \otimes 1^o \circ x = 1 \otimes a^o \circ x$ for all $a \in A\}$, i.e. $S(A)$ is the centralizer of the subalgebra A in $A\#(kG)^*$. It was shown in [1, p. 688] that $S(A)$ is a Galois extension for G abelian. The proof also holds for nonabelian G; we include it here for the sake of completeness.

Since $S(A)^G = (A\#(kG)^*)^G \cap S(A) = A^A = k$, it remains to show that the map Γ: $S(A) \otimes S(A) \to S(A) \otimes (kG)^*$ is an isomorphism. Note that since A is an Azumaya algebra and $A\#(kG)^*$ is a left $A \otimes A^o$-module, the map m: $A \otimes S(A) = A \otimes (A\#(kG)^*)^A \to A\#(kG)^*$ defined by $m(a \otimes x) = ax$ is an isomorphism [7, 2.13]. It is easy to check that the map η: $(A\#(kG)^*) \otimes_A (A\#(kG)^*) \to (A\#(kG)^*) \otimes (kG)^*$ defined by $\eta((a\#p_\sigma) \otimes_A (b\#p_\tau)) = \sum_{\alpha \in G} (ab_\alpha \# p_{\alpha^{-1}\sigma}) \otimes p_{\sigma^{-1}\alpha\tau}$ is bijective with inverse η^{-1} defined by $\eta^{-1}((a\#p_\sigma) \otimes p_\tau) = (a\#p_\sigma) \otimes_A (1\#p_{\sigma\tau})$. The following diagram commutes (cf. [4, p. 310]).

$$
\begin{tikzcd}
A \otimes S(A) \otimes S(A) \arrow[r, "1\otimes\Gamma"] \arrow[d, "m\otimes 1"'] & A \otimes S(A) \otimes (kG)^* \arrow[ddd, "m\otimes 1"] \\
(A\#(kG)^*) \otimes S(A) \arrow[d, "\cong"'] & \\
(A\#(kG)^*) \otimes_A A \otimes S(A) \arrow[d, "1\otimes m"'] & \\
(A\#(kG)^*) \otimes_A (A\#(kG)^*) \arrow[r, "\eta"] & A\#(kG)^*) \otimes (kG)^*
\end{tikzcd}
$$

Since all the maps but $1 \otimes \Gamma$ are already known to be isomorphisms, $1 \otimes \Gamma$ is an isomorphism and since A is faithfully flat, Γ is also. Thus, $S(A)$ is indeed a Galois extension with group G.

Theorem. *The following statements are equivalent.*

(i) *The G-grading on A is inner.*

(ii) *The Galois extension $S(A) \supseteq k$ has a normal basis.*

(iii) *There exist $x, y \in S(A)$ such that $\sum_{\sigma \in G} \sigma(x)\tau\sigma(y) = \delta_{\tau,1}$.*

Proof. We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Suppose first that the G-grading on A is inner, i.e. there exist maps u and v from G to A such that for all $a \in A$, $\gamma \in G$, $a_\gamma = \sum_{\alpha\in G} u(\gamma\alpha)av(\alpha^{-1})$. Define a k-module map Ω from $(kG)^*$ to $S(A)$ by $\Omega(\beta^{-1}(p_1)) = \Omega(p_\beta) = \sum_{\alpha\in G} v(\alpha^{-1})\#p_{\alpha\beta} = \beta^{-1}(\Omega(p_1))$. Note that $\Omega(p_1)$, and therefore the image of Ω, is in $S(A) = (A\#(kG)^*)^A$ since for $c \in A$,

$$
\begin{aligned}
\left(\sum_{\alpha\in G} v(\alpha^{-1})\#p_\alpha\right)(c\#1) &= \sum_{\alpha,\tau\in G} v(\alpha^{-1})c_{\alpha\tau^{-1}}\#p_\tau \\
&= \sum_{\alpha,\tau,\sigma\in G} v(\alpha^{-1})u(\alpha\tau^{-1}\sigma)cv(\sigma^{-1})\#p_\tau \\
&= \sum_{\sigma,\tau\in G}\left\{\sum_{\alpha\in G} v(\alpha^{-1})u(\alpha\tau^{-1}\sigma)\right\}cv(\sigma^{-1})\#p_\tau \\
&= \sum_{\sigma,\tau\in G}\{\delta_{1,\tau^{-1}\sigma}\}cv(\sigma^{-1})\#p_\tau \qquad \text{by (2)} \\
&= \sum_{\tau\in G} cv(\tau^{-1})\#p_\tau \\
&= (c\#1)\sum_{\alpha\in G} v(\alpha^{-1})\#p_\alpha .
\end{aligned}
$$

Therefore for all $c \in A$,

$$
(*) \qquad cv(\gamma) = \sum_{\beta\in G} v(\beta^{-1})c_{\beta\gamma} = \sum_{\delta\in G} v(\gamma\delta^{-1})c_\delta .
$$

It remains to show that Ω is an isomorphism.

Suppose $\Omega(\sum_{\sigma\in G} r(\sigma)p_\sigma) = \sum_{\sigma,\alpha\in G} r(\sigma)v(\sigma\alpha^{-1})\#p_\alpha = 0$ for scalars $r(\sigma) \in k$, $\sigma \in G$. Then $\sum_{\sigma\in G} r(\sigma)v(\sigma\alpha^{-1}) = 0$ for all $\alpha \in G$. Then, for any $\lambda \in G$,

$$
0 = \sum_{\sigma\in G} r(\sigma)v(\sigma\alpha^{-1})u(\alpha\lambda^{-1}),
$$

so that

$$
\begin{aligned}
0 &= \sum_{\alpha\in G}\sum_{\sigma\in G} r(\sigma)v(\sigma\alpha^{-1})u(\alpha\lambda^{-1}) \\
&= \sum_{\sigma\in G} r(\sigma)\left\{\sum_{\rho\in G} v(\sigma\lambda^{-1}\rho^{-1})u(\rho)\right\} \qquad \text{where } \rho = \alpha\lambda^{-1} \\
&= r(\lambda) \qquad \text{by (2)}.
\end{aligned}
$$

Thus Ω is one-one. Locally Ω is an isomorphism since locally $S(A)$ is free of rank equal to the order of G; therefore Ω is an isomorphism.

(ii) $\Rightarrow$ (iii). This implication follows from [8, Lemma 2].

(iii) $\Rightarrow$ (i). We suppose next that there exist $x, y \in S(A)$ such that $\sum_{\sigma\in G}\sigma(x)\tau\sigma(y) = \delta_{\tau,1}$. Let $x = \sum_{\alpha\in G} a(\alpha)\#p_\alpha$, $y = \sum_{\beta\in G} b(\beta)\#p_\beta$. Then

$$\begin{aligned}
\delta_{\tau,1} &= \sum_{\sigma\in G}\sigma(x)\tau\sigma(y)\\
&= \sum_{\sigma,\beta\in G}(\sigma(x))(b(\beta)\#p_{\beta(\tau\sigma)^{-1}})\\
&= \sum_{\sigma,\beta\in G}(b(\beta)\#1)\sigma(x)(1\#p_{\beta(\tau\sigma)^{-1}}) \qquad \text{since } \sigma(x)\in S(A)\\
&= \sum_{\alpha,\beta,\sigma\in G}(b(\beta)a(\alpha)\#p_{\alpha\sigma^{-1}})(1\#p_{\beta(\tau\sigma)^{-1}})\\
&= \sum_{\beta,\sigma\in G} b(\beta)a(\beta(\tau\sigma)^{-1}\sigma)\#p_{\beta(\tau\sigma)^{-1}}\\
&= \sum_{\beta,\lambda\in G} b(\beta)a(\lambda\tau^{-1}\lambda^{-1}\beta)\#p_\lambda \qquad \text{with } \lambda = \beta(\tau\sigma)^{-1}
\end{aligned}$$

and since the coefficient of p_λ is $\delta_{\tau,1}$ for all λ, we have

$$(**) \qquad \sum_{\beta\in G} b(\beta)a(\eta\beta) = \delta_{\eta,1} \qquad \text{for all } \eta\in G.$$

Now for any $c\in A$,

$$\begin{aligned}
c_\tau &= \sum_{\gamma\in G}\{\delta_{\tau,\gamma}\}c_\gamma\\
&= \sum_{\gamma\in G}\left\{\sum_{\rho\in G} b(\tau\rho)a(\gamma\rho)\right\}c_\gamma \qquad \text{by } (**)\\
&= \sum_{\rho\in G} b(\tau\rho)\left\{\sum_{\gamma\in G} a(\gamma\rho)c_\gamma\right\}\\
&= \sum_{\rho\in G} b(\tau\rho)ca(\rho) \qquad \text{since } x\in S(A).
\end{aligned}$$

Now define maps u, v from G to A by $u(\sigma) = b(\sigma)$ and $v(\tau^{-1}) = a(\tau)$. □

This completes the proof of the main theorem. Note that in the proof that (iii) $\Rightarrow$ (i), the fact that y is in $S(A)$ is not used. In fact, condition (iii) is equivalent to the following:

(iii)$'$ There exist $x\in S(A)$, $y\in A\#(kG)^*$ such that $\sum_{\sigma\in G}\sigma(x)\tau\sigma(y) = \delta_{\tau,1}$.

Corollary 1. *If* MSpec(k), *the maximal ideal spectrum of* k, *has a basis of open/closed sets, (for example if* k *is semilocal or von Neumann regular), then every* G*-graded* k*-Azumaya algebra has inner grading.*

Proof. It is proved in [6] that if MSpec(k) has a basis of open/closed sets, and H is a finitely generated commutative or cocommutative Hopf algebra, then every Galois H-object has normal basis. Let H be the commutative Hopf algebra $(kG)^*$. □

Corollary 2. *If every G-graded k-Azumaya algebra has inner grading, then every $(kG)^*$-Galois object has normal basis.*

Proof. The statement follows from the fact that every Galois $(kG)^*$-object is isomorphic to $S(A)$ for some graded Azumaya algebra A. The proof is essentially the same as [1, p. 689–90]. We outline the argument.

If S is a Galois extension of k with group G, then $A = S\#kG$ is a G-graded Azumaya algebra with $A_\sigma = S\#\sigma$, $\sigma \in G$. Then

$$\phi\colon S(A) = ((S\#kG)\#(kG)^*)^{(S\#kG)} \to S$$

defined by $\phi(\sum_{\sigma,\tau\in G}(s(\sigma,\tau)\#\sigma)\#p_\tau) = \sum_{\sigma\in G} s(\sigma, 1)$ is a kG-algebra homomorphism. But a kG-algebra homomorphism between two Galois extensions is an isomorphism. □

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