PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 4, December 1989

A SIMPLE PROOF OF LIVINGSTON'S INEQUALITY FOR CARATHÉODORY FUNCTIONS

PHILIPPE DELSARTE AND YVES GENIN

(Communicated by Irwin Kra)

ABSTRACT. The Livingston determinant inequality involving the Maclaurin coefficients of a Carathéodory function are derived in a straightforward manner by use of the Riesz-Herglotz representation and the Schwarz inequality. The result is extended to the case of matrix-valued functions.

1. INTRODUCTION

Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be a function of the complex variable z, analytic in the unit disc $\mathbf{D} = \{z : |z| < 1\}$ and satisfying

$$\operatorname{Re}(f(z)) \geq 0 \qquad \text{for } z \in \mathbf{D}. \tag{1}$$

Then $f(z)$ is called a *Carathéodory function.* The theory of this class of functions has numerous applications in pure and applied mathematics. An introduction to the subject can be found in Akhiezer's classical book [1]. In the sequel it is assumed that $f(z)$ does not reduce to an imaginary constant or, equivalently, that the real part of f_0 is strictly positive.

Consider the reciprocal function $g(z) = f(z)^{-1} = \sum_{k=0}^{\infty} g_k z^k$. (Note that $g(z)$ is a Carathéodory function.) For any nonnegative integer n, let $g_n(z) = \sum_{k=0}^{n} g_k z^k$ denote the order n truncation polynomial of $g(z)$. Define the function $v_n(z) = \sum_{k=0}^{\infty} v_{n,k} z^k$, analytic at the origin, through the identity

$$g_n(z) f(z) = 1 + 2z^{n+1} v_n(z). \tag{2}$$

Motivated by a coefficient problem for a certain class of multivalent functions, Livingston has recently established the following remarkable inequality [4]:

$$|v_{n,k}| \leq |f_0|^{-1} \operatorname{Re}(f_0), \tag{3}$$

for all nonnegative integers n and k.

Received by the editors March 21, 1988 and, in revised form, February 27, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D50; Secondary 30E05.
Key words and phrases. Carathéodory functions, coefficient inequalities, Riesz-Herglotz representation.

As shown in [4], the number $v_{n,k}$ can be expressed in terms of the determinant of the Hessenberg matrix

$$F_{n,k} = \begin{bmatrix} f_1 & f_2 & \cdots & f_n & f_{n+k+1} \\ f_0 & f_1 & \cdots & f_{n-1} & f_{n+k} \\ & f_0 & \cdots & f_{n-2} & f_{n+k-1} \\ & & \ddots & \vdots & \vdots \\ & & & f_0 & f_{k+1} \end{bmatrix} \tag{4}$$

built from the Maclaurin coefficients f_m of the given function $f(z)$. In fact, it is easily seen that $v_{n,k}$ is given by

$$v_{n,k} = (-1)^n \frac{\det(F_{n,k})}{2f_0^{n+1}}. \tag{5}$$

This follows from the fact that (2) contains the system of linear relations

$$[g_0, g_1, \ldots, g_{n-1}, g_n]F_{n,k} = [0, 0, \ldots, 0, 2v_{n,k}]. \tag{6}$$

Solving (6) for the first "unknown", g_0, we obtain (5) by use of $g_0 = f_0^{-1}$. Livingston's argument starts from the explicit formula (5), uses a suitable approximation of $f(z)$ by a rational singular Carathéodory function, involves some manipulations of Hessenberg determinants, and produces the result (3) by means of the Cauchy inequality [4].

This paper gives a more direct proof of Livingston's result (3), based on the integral representation of $v_{n,k}$ that follows from the Riesz-Herglotz formula for the Carathéodory function $f(z)$. In this setting, the Schwarz inequality leads immediately to the desired result. After explaining the proof in some detail, we briefly examine the case where the bounds (3) are sharp. A final section contains an extension of Livingston's result to matrix-valued Carathéodory functions.

2. Proof

Let us denote by $d\sigma$ the positive measure that corresponds to the Carathéodory function $f(z)$ via the *Riesz-Herglotz integral representation* [1], [3]. The relation is given by

$$f(z) = i\,\mathrm{Im}(f_0) + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta). \tag{7}$$

Roughly speaking, the Maclaurin coefficients of $f(z)$ are the trigonometric moments of $d\sigma$. More precisely, (7) is equivalent to the set of formulas

$$\mathrm{Re}(f_0) = \int_0^{2\pi} d\sigma(\theta), \tag{8}$$

$$f_m = 2\int_0^{2\pi} e^{-im\theta}\, d\sigma(\theta) \qquad \text{for } 1 \le m < \infty. \tag{9}$$

The Livingston number $v_{n,k}$ defined from (2) can be expressed in the form

$$v_{n,k} = \frac{1}{2}\sum_{l=0}^{n} g_l f_{n+k+1-l}. \tag{10}$$

(This is the last component of the system (6).) Using (9) we then obtain the key formula

$$v_{n,k} = \int_0^{2\pi} e^{-i(n+k+1)\theta} g_n(e^{i\theta})\, d\sigma(\theta). \tag{11}$$

Applying the Schwarz inequality yields

$$|v_{n,k}|^2 \le \int_0^{2\pi} d\sigma(\theta) \int_0^{2\pi} |g_n(e^{i\theta})|^2\, d\sigma(\theta). \tag{12}$$

Elementary computation shows that (12) amounts to the desired result (3), written in the form

$$|v_{n,k}|^2 \le \mathrm{Re}(f_0)\,\mathrm{Re}(f_0^{-1}). \tag{13}$$

Indeed, in view of (8) and (9) we can write the identity

$$\int_0^{2\pi} |g_n(e^{i\theta})|^2\, d\sigma(\theta) = \frac{1}{2}\mathbf{g}_n(F_n + F_n^*)\mathbf{g}_n^*, \tag{14}$$

with $\mathbf{g}_n = [g_0, g_1, \ldots, g_n]$ and

$$F_n = \begin{bmatrix} f_0 & f_1 & \cdots & f_n \\ & f_0 & \cdots & f_{n-1} \\ & & \ddots & \vdots \\ & & & f_0 \end{bmatrix}. \tag{15}$$

Since $\mathbf{g}_n F_n = [1, 0, \ldots, 0]$, by (2), the right hand side of (14) equals $\mathrm{Re}(g_0)$. This proves the claim (13), by use of (8) and of $g_0 = f_0^{-1}$.

The argument above allows us to discuss the "case of equality" in an efficient manner. Indeed, it is seen that (3) is sharp, for given values of n and k, if and only if the polynomial $g_n(z)$ is proportional to the monomial z^{n+k+1} on the support of the measure $d\sigma$ (viewed as a subset of the unit circle $|z| = 1$). Without going into details, let us mention the following consequence of this fact. For a given n, equality holds in (3) for two successive values of k if and only if the support of $d\sigma$ reduces to a single point; this means that $f(z)$ is a singular Carathéodory function of degree 1, i.e., a function of the form

$$f(z) = i\beta + \alpha\frac{e^{i\gamma} + z}{e^{i\gamma} - z}, \tag{16}$$

where α, β and γ are real numbers, with $\alpha > 0$. In this case, the bound is sharp for all nonnegative integers n and k.

3. Generalization

Let us now briefly examine how the Livingston inequality can be generalized to the case of a *matrix-valued Carathéodory function.* The definition is formally the same as in §1, except that the coefficients f_k are square matrices of any fixed order p. The condition (1) means that the Hermitian part (or "real part") of $f(z)$ is nonnegative definite in the unit disc. In addition, we assume here that $f(z)$ is *nondegenerate*, in the sense that the matrix $\mathrm{Re}(f_0)$ is invertible.

Define the $p \times p$ matrix polynomial $g_n(z)$, of formal degree n, and the $p \times p$ matrix function $v_n(z)$ from the identity (2), where 1 is interpreted as the identity matrix. It can be shown that the matrix coefficients $v_{n,k}$ of $v_n(z)$ satisfy the inequality

$$v_{n,k}v_{n,k}^* \leq \mathrm{Re}(\mathrm{tr}(f_0))\,\mathrm{Re}(f_0^{-1}), \tag{17}$$

generalizing (15). Here, the star denotes the conjugate transpose, Re is the Hermitian part, tr stands for the trace (i.e., the sum of diagonal entries), and the inequality $a \leq b$ means that $b - a$ is nonnegative definite.

The proof of (17) is essentially the same as in §2. It is based on the Riesz-Herglotz representation (7) for nondegenerate matrix-valued Carathéodory functions and makes use of a suitable matrix extension of the Schwarz inequality [2]. Details will not be given.

References

1. N. I. Akhiezer, *The classical moment problem*, Oliver and Boyd, London, 1965.
2. P. Delsarte, Y. Genin and Y. Kamp, *Orthogonal polynomial matrices on the unit circle*, IEEE Trans. Circuits and Systems **CAS-25** (1978), 149–160.
3. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
4. A. E. Livingston, *A coefficient inequality for functions of positive real part with an application to multivalent functions*, Pacific J. Math. **120** (1985), 139–151.

Philips Research Laboratory, Brussels, Belgium