A SIMPLE PROOF OF LIVINGSTON'S INEQUALITY FOR CARATHÉODORY FUNCTIONS

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Abstract. The Livingston determinant inequality involving the Maclaurin coefficients of a Carathéodory function are derived in a straightforward manner by use of the Riesz-Herglotz representation and the Schwarz inequality. The result is extended to the case of matrix-valued functions.

1. Introduction

Let \( f(z) = \sum_{k=0}^{\infty} f_k z^k \) be a function of the complex variable \( z \), analytic in the unit disc \( D = \{ z : |z| < 1 \} \) and satisfying
\[
\text{Re}(f(z)) \geq 0 \quad \text{for } z \in D.
\]
Then \( f(z) \) is called a Carathéodory function. The theory of this class of functions has numerous applications in pure and applied mathematics. An introduction to the subject can be found in Akhiezer's classical book [1]. In the sequel it is assumed that \( f(z) \) does not reduce to an imaginary constant or, equivalently, that the real part of \( f_0 \) is strictly positive.

Consider the reciprocal function \( g(z) = f(z)^{-1} = \sum_{k=0}^{\infty} g_k z^k \). (Note that \( g(z) \) is a Carathéodory function.) For any nonnegative integer \( n \), let \( g_n(z) = \sum_{k=0}^{n} g_k z^k \) denote the order \( n \) truncation polynomial of \( g(z) \). Define the function \( v_n(z) = \sum_{k=0}^{\infty} v_{n,k} z^k \), analytic at the origin, through the identity
\[
g_n(z)f(z) = 1 + 2z^{n+1}v_n(z).
\]
Motivated by a coefficient problem for a certain class of multivalent functions, Livingston has recently established the following remarkable inequality [4]:
\[
|v_{n,k}| \leq |f_0|^{-1}\text{Re}(f_0),
\]
for all nonnegative integers \( n \) and \( k \).
As shown in [4], the number $v_{n,k}$ can be expressed in terms of the determinant of the Hessenberg matrix

$$F_{n,k} = \begin{bmatrix} f_1 & f_2 & \cdots & f_n & f_{n+k+1} \\ f_0 & f_1 & \cdots & f_{n-1} & f_{n+k} \\ f_0 & \cdots & f_{n-2} & f_{n+k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_0 & \cdots & \cdots & \cdots & f_{k+1} \end{bmatrix}$$

built from the Maclaurin coefficients $f_m$ of the given function $f(z)$. In fact, it is easily seen that $v_{n,k}$ is given by

$$v_{n,k} = (-1)^n \frac{\det(F_{n,k})}{2f_0^{n+1}}.$$ 

This follows from the fact that (2) contains the system of linear relations

$$[g_0, g_1, \ldots, g_{n-1}, g_n]F_{n,k} = [0, 0, \ldots, 0, 2v_{n,k}].$$

Solving (6) for the first "unknown", $g_0$, we obtain (5) by use of $g_0 = f_0^{-1}$. Livingston's argument starts from the explicit formula (5), uses a suitable approximation of $f(z)$ by a rational singular Carathéodory function, involves some manipulations of Hessenberg determinants, and produces the result (3) by means of the Cauchy inequality [4].

This paper gives a more direct proof of Livingston's result (3), based on the integral representation of $v_{n,k}$ that follows from the Riesz-Herglotz formula for the Carathéodory function $f(z)$. In this setting, the Schwarz inequality leads immediately to the desired result. After explaining the proof in some detail, we briefly examine the case where the bounds (3) are sharp. A final section contains an extension of Livingston's result to matrix-valued Carathéodory functions.

2. Proof

Let us denote by $d\sigma$ the positive measure that corresponds to the Carathéodory function $f(z)$ via the Riesz-Herglotz integral representation [1], [3]. The relation is given by

$$f(z) = i \text{Im}(f_0) + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta).$$

Roughly speaking, the Maclaurin coefficients of $f(z)$ are the trigonometric moments of $d\sigma$. More precisely, (7) is equivalent to the set of formulas

$$\text{Re}(f_0) = \int_0^{2\pi} d\sigma(\theta),$$

$$f_m = 2 \int_0^{2\pi} e^{-im\theta} d\sigma(\theta) \quad \text{for } 1 \leq m < \infty.$$
The Livingston number $v_{n,k}$ defined from (2) can be expressed in the form

$$v_{n,k} = \frac{1}{2} \sum_{l=0}^{n} g_l f_{n+k+1-l}. \tag{10}$$

(This is the last component of the system (6).) Using (9) we then obtain the key formula

$$v_{n,k} = \int_{0}^{2\pi} e^{-i(n+k+1)\theta} g_n(e^{i\theta}) d\sigma(\theta). \tag{11}$$

Applying the Schwarz inequality yields

$$|v_{n,k}|^2 \leq \int_{0}^{2\pi} d\sigma(\theta) \int_{0}^{2\pi} |g_n(e^{i\theta})|^2 d\sigma(\theta). \tag{12}$$

Elementary computation shows that (12) amounts to the desired result (3), written in the form

$$|v_{n,k}|^2 \leq \text{Re}(f_{0}) \text{Re}(f_{0}^{-1}). \tag{13}$$

Indeed, in view of (8) and (9) we can write the identity

$$\int_{0}^{2\pi} |g_n(e^{i\theta})|^2 d\sigma(\theta) = \frac{1}{2} g_n(F_n + F_n^*) g_n^*, \tag{14}$$

with $g_n = [g_0, g_1, \ldots, g_n]$ and

$$F_n = \begin{bmatrix} f_0 & f_1 & \cdots & f_n \\ f_0 & f_1 & \cdots & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_0 & & & f_0 \end{bmatrix}. \tag{15}$$

Since $g_n F_n = [1, 0, \ldots, 0]$, by (2), the right hand side of (14) equals $\text{Re}(g_0)$. This proves the claim (13), by use of (8) and of $g_0 = f_0^{-1}$.

The argument above allows us to discuss the "case of equality" in an efficient manner. Indeed, it is seen that (3) is sharp, for given values of $n$ and $k$, if and only if the polynomial $g_n(z)$ is proportional to the monomial $z^{n+k+1}$ on the support of the measure $d\sigma$ (viewed as a subset of the unit circle $|z| = 1$). Without going into details, let us mention the following consequence of this fact. For a given $n$, equality holds in (3) for two successive values of $k$ if and only if the support of $d\sigma$ reduces to a single point; this means that $f(z)$ is a singular Carathéodory function of degree 1, i.e., a function of the form

$$f(z) = i\beta + \alpha \frac{e^{i\gamma} + z}{e^{i\gamma} - z}, \tag{16}$$

where $\alpha$, $\beta$ and $\gamma$ are real numbers, with $\alpha > 0$. In this case, the bound is sharp for all nonnegative integers $n$ and $k$. 

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3. Generalization

Let us now briefly examine how the Livingston inequality can be generalized to the case of a matrix-valued Carathéodory function. The definition is formally the same as in §1, except that the coefficients $f_k$ are square matrices of any fixed order $p$. The condition (1) means that the Hermitian part (or "real part") of $f(z)$ is nonnegative definite in the unit disc. In addition, we assume here that $f(z)$ is nondegenerate, in the sense that the matrix $\text{Re}(f_0)$ is invertible.

Define the $p \times p$ matrix polynomial $g_n(z)$, of formal degree $n$, and the $p \times p$ matrix function $v_n(z)$ from the identity (2), where $1$ is interpreted as the identity matrix. It can be shown that the matrix coefficients $v_{n,k}$ of $v_n(z)$ satisfy the inequality

$$v_{n,k} v_{n,k}^* \leq \text{Re}(\text{tr}(f_0)) \text{Re}(f_0^{-1})$$

generalizing (15). Here, the star denotes the conjugate transpose, $\text{Re}$ is the Hermitian part, $\text{tr}$ stands for the trace (i.e., the sum of diagonal entries), and the inequality $a \leq b$ means that $b - a$ is nonnegative definite.

The proof of (17) is essentially the same as in §2. It is based on the Riesz-Herglotz representation (7) for nondegenerate matrix-valued Carathéodory functions and makes use of a suitable matrix extension of the Schwarz inequality [2]. Details will not be given.

References


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