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ON THE ZEROS OF $L' + L^2$ FOR CERTAIN RATIONAL FUNCTIONS L

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ABSTRACT. Let L be a nonconstant rational function whose poles are real, simple with each one having a positive residue. Then, if $L' + L^2$ has no nonreal zeros, L has the form

$$L(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z - x_k} - az + b,$$

x_k are real, $\alpha_k > 0$ for $1 \leq k \leq n$, $a \geq 0$ and b is real. In particular, if P is a polynomial of degree ≥ 2, then $P' + P^2$ has nonreal zeros. The result is applied to entire functions in connection with zeros of the derivatives.

It is well known that if f is a real entire function in the Laguerre-Pólya class, then f, f', f'', and all higher derivatives of f have isolated zeros only on the real axis. The converse problem—that only functions in the Laguerre-Pólya class have this property—was proposed by Pólya in 1914, but no general solution was found until the work of Hellerstein and Williamson [1, 2] in 1977. They showed that it is sufficient to know that f, f', and f'' have only real zeros in order to deduce that f lies in the Laguerre-Pólya class. Nevertheless as early as 1915 A. Wiman in his work on real entire functons of finite order possessing only real zeros, had proposed a delicate conjecture relating the order of the entire function to the number of nonreal zeros of f''. In particular Wiman's conjecture implies that if f is a real entire function of finite order such that f and f'' have only real zeros, then f is in the Laguerre-Pólya class. Recently the author has given a proof of Wiman's conjecture [4]. Among the possible directions for generalisation, there are two which immediately suggest themselves: (I) to remove the hypothesis of finite order; (II) to remove the hypothesis that the functions be real on the real axis. In this note we shall consider a special case of (II), namely when f has finite order and possesses only a finite number of zeros, each of which is real. The logarithmic derivatives of such functions are rational, which enables us to apply algebraic arguments. Theorem 1 gives our main result and Theorem 2 the application to entire functions.

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Theorem 1. *Let L be a nonconstant rational function whose poles (if any) are real and simple with each one having a positive residue. Then $L' + L^2$ has nonreal zeros except in the case that L is real on the real axis (except at the poles) and* $\operatorname{Im} L(z) < 0$ *for* $\operatorname{Im} z > 0$.

The exceptional functions L are those which can be written in the form

$$L(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z - x_k} - az + b, \tag{1}$$

where x_k are real, $\alpha_k > 0$ for $1 \le k \le n$, $a \ge 0$, and b is real.

This result has recently been established by the author [4] under the additional hypothesis that L is a real rational function (i.e. real on the real axis except at the poles). Indeed, the conclusion of the theorem is valid for a real meromorphic function L, whose poles are real, simple, and with positive residues, provided that L satisfies the growth condition

$$m(r, L) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |L(re^{i\theta})|\, d\theta = O(\log r) \qquad \text{as } r \to \infty. \tag{2}$$

From Theorem 1 we immediately deduce

Theorem 2. *Let f be an entire function of finite order possessing at most a finite number of zeros each of which is real. Suppose that $f''(z)$ has no nonreal zeros. Then either* (i) *$f(z) = e^{az+b}$, where a and b are constants, or* (ii) *there exists a real constant θ such that $f(z) = e^{i\theta} g(z)$, where g is a real entire function in the Laguerre-Pólya class. In particular, f' and all higher derivatives of f have no nonreal isolated zeros.*

Proof. Let $L = f'/f$ and apply Theorem 1.

Lemma 1. *Every rational function R can be written uniquely in the form $R = U + iV$, where U and V are real rational functions.*

Proof. Let $U(z) = \frac{1}{2}(R(z) + \overline{R}(\overline{z}))$, $V(z) = -\frac{1}{2}i(R(z) - \overline{R}(\overline{z}))$.

Lemma 2. *Let R be a rational function all of whose zeros and poles are real. Then there exists a real constant θ such that $R(z) = e^{i\theta} S(z)$, where S is a real rational function.*

Proof. Let S be a real rational function having the same zeros and poles as R with the same orders, e.g. $S = P/Q$, where P and Q are monic polynomials with real zeros. Then R/S is a rational function with no zeros nor poles, so is a nonzero constant by the fundamental theorem of algebra. Hence $R = \rho e^{i\theta} S$ and ρS is a real rational function.

Lemma 3. *Let L be a nonconstant rational function whose poles are real, simple, and have positive residues. Suppose that the function $L' + L^2$ has no nonreal zeros. Then L is a real rational function.*

Proof. Writing $L = U + iV$, where U and V are real rational functions, we see that U has the same poles as L, of simple order and with positive residues, whereas V has zero residues at these poles, so is a polynomial. We have that

$$L' + L^2 = U' + U^2 - V^2 + i(V' + 2UV) \tag{3}$$

has all its zeros and poles real, and so from Lemma 2 there exists a real constant θ such that

$$(U' + U^2 - V^2)\sin\theta = (V' + 2UV)\cos\theta. \tag{4}$$

Suppose that U has a pole at x with residue α, so that

$$U(z) = \frac{\alpha}{z - x} + \omega(z) \tag{5}$$

where ω is regular at x. Then

$$\lim_{z \to x}(z - x)^2(U^2(z) + U'(z)) = \alpha^2 - \alpha \tag{6}$$

and hence $\alpha(\alpha - 1)\sin\theta = 0$. Thus either (i) $\sin\theta = 0$ or (ii) $\alpha = 1$.

Case (i). $\sin\theta = 0$. Then $V' + 2UV = 0$, and so either $V = 0$ and L is real, or $2U = -V'/V$. If $V' = 0$, then $U = 0$ and L is constant. Otherwise V is a nonconstant polynomial whose zeros are the poles of U and the residues at these poles are negative, contradicting the hypothesis. Thus Case (i) implies that L is real, as required.

Case (ii). $\sin\theta \neq 0$ and $\alpha = 1$ is the residue at each pole of U. Then we can write

$$U = \frac{\varphi'}{\varphi} + P \tag{7}$$

where φ is a real polynomial, not identically zero, with simple real zeros, and where P is a real polynomial. Substituting into (4) we obtain

$$(\varphi'' + 2P\varphi' + (P' + P^2 - V^2)\varphi)\sin\theta = (2V\varphi' + (V' + 2VP)\varphi)\cos\theta. \tag{8}$$

This equation may be rewritten

$$\begin{aligned} &\varphi(P\sin\theta - V(\cos\theta + 1))(P\sin\theta - V(\cos\theta - 1)) \\ &\quad = \sin\theta\cos\theta(2V\varphi' + V'\varphi) - \sin^2\theta(\varphi'' + 2P\varphi' + P'\varphi). \end{aligned} \tag{9}$$

Writing $d(\psi)$ for the degree of a polynomial ψ, we note that the degree of the right-hand side of this equation is at most $d(\varphi) + \max(d(P), d(V)) - 1$. On the other hand, unless the left-hand side is identically zero, the degree of the left-hand side is at least $d(\varphi) + \max(d(P), d(V))$. Thus both sides of the equation vanish identically and we have

$$P\sin\theta = V(\cos\theta \pm 1), \tag{10}$$

$$\cos\theta(2V\varphi' + V'\varphi) - \varphi''\sin\theta - 2V\varphi'(\cos\theta \pm 1) - V'\varphi(\cos\theta \pm 1) = 0. \tag{11}$$

This last equation reduces to

$$\varphi\varphi''\sin\theta = \pm(\varphi^2 V)'. \tag{12}$$

The degrees of the polynomials on each side of this equation are different, unless both sides vanish identically. It follows that both φ and V are constant, and therefore applying (7) and (10) we deduce that L is constant. This completes the proof of the lemma.

As Lemma 3 reduces the proof of Theorem 1 to the case when L is real, the result follows from [4, Theorem 2].

Corollary. *Let P be a polynomial of degree ≥ 2. Then $P' + P^2$ has at least one nonreal zero.*

When P is a real polynomial, $P' + P^2$ has at least $d(P) - 1$ nonreal zeros [4]. It seems likely that this lower bound will hold generally, and indeed we would expect [4, Theorem 2] to hold without the assumption that L is real.

It would certainly be of interest to generalise the results of Theorems 1 and 2 to general meromorphic and entire functions not assumed to be real. For an account of progress in this area we refer to the papers of Hellerstein, Shen and Williamson [3], [5].

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