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NOTES OF THE INVERSION OF INTEGRALS I

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ABSTRACT. If W is a Picard bundle on the Jacobian J of a curve C, we have the problem of describing W globally. The theta divisor θ is ample on J. Thus it is possible to write n^*W as the sheaf associated to a graded M over the well-known ring $\oplus_{m\geq 0}\Gamma(J, \mathscr{O}_J(m4\theta))$. In this paper we compute the degree of generators and relations for such a module M.

There are naturally occurring locally free sheaves called Picard bundles on the Jacobian J of a smooth complete curve C of positive genus g over $k = \overline{k}$. These bundles describe the global variation of the sections of invertible sheaves on C with pleasant degree.

The inversion problem is to give a description of the Picard bundles globally on J. As such analytic description is lacking, we must content ourselves with two algebraic solutions of this problem.

The first solution requires us to know the image of some points of C in the Jacobian. This approach uses a method due to R. C. Gunning. The second solution determines the pull-back of the Picard bundle by a multiplication in J in terms of a module over the graded ring of theta sections. Here one uses a form of a theorem of D. Mumford on the equations defining abelian varieties projectively.

1. THE FIRST METHOD

Let $\mathscr{P}$ be a Poincaré sheaf on $J \times J$. Let $\mathscr{L}_n$ be an invertible sheaf on J of the form $\mathscr{O}_j(n\theta)$ where the divisor θ gives the usual principal polarization of J. If $n > 0$ then $\pi_2^*\mathscr{L}_n \otimes \mathscr{P}$ is a family of ample invertible sheaves on the second factor. It follows from Mumford's vanishing theorem that

$$R^i\pi_{1*}(\pi_2^*\mathscr{L}_n \otimes \mathscr{P})$$

is zero if $i > 0$ and $\mathscr{V}_n = \pi_{1*}(\pi_2^*\mathscr{L}_n \otimes \mathscr{P})$ is a locally free sheaf of rank n^g.

Let $C \hookrightarrow J$ be a universal abelian integral. Let $Q_n = \pi_2^*\mathscr{L}_n \otimes \mathscr{P}|_{J\times C}$. Then Q_n is a family of invertible sheaves on C of degree $n \cdot g$ as $[C\colon \theta] = g$. If

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$n > 1$ then $R^i\pi_{J*}Q^n$ is zero if $i > 0$ and the Picard sheaf $\mathscr{W}_{ng} = \pi_{J*}Q_n$ is locally free of rank $ng - g + 1 = (n-1)g + 1$.

The inclusion of C in J induces a restriction homomorphism $\alpha\colon \mathscr{U}_n \to \mathscr{W}_{ng}$. The critical fact is

Proposition 1 (Gunning [2]). *α is surjective if $n > 1$.*

Let D be an effective divisor on C of degree d. Then $Q_n(-J \times D)$ is a family of invertible sheaves on C of degree $n \cdot g - d$. If $d > n \cdot g$, then $\pi_{J*}(Q_n(-J \times D))$ is zero and the Picard sheaf $\mathscr{U}_n(D) \equiv R^1_{J*}(Q_n(-J \times D))$ is locally free of rank $d - n \cdot g + g - 1 = d - (n-1)g - 1$. Consider the exact sequence

$$0 \to Q_n(-J \times D) \hookrightarrow Q_n \to Q_n|_{J \times D} \to 0.$$

This yields the well-known exact sequence of

Lemma 2. *We have an exact sequence*

$$0 \to \mathscr{W}_{ng} \xrightarrow{e} \pi_{J*}(Q_n|_{J\times D}) \to \mathscr{U}_n(D) \to 0$$

where e is just evaluation.

The composition $\beta_D\colon \mathscr{V}_n \xrightarrow{\alpha} \mathscr{W}_{ng} \xrightarrow{e} \pi_{J*}(Q_n|_{J\times D}) = \pi_{J*}(\pi_2^*\mathscr{L}_n \otimes \mathscr{P}|_{J\times D})$ is simply restriction and is determined only by how D sits as a closed subscheme of J. The combination of the above facts give the first solution of the inversion problem.

Theorem 3. *$\mathscr{W}_{ng} = \mathrm{Image}(\beta_D)$ and $\mathscr{U}_n(D) = \mathrm{Cokernel}(\beta_D)$.*

2. Normal presentation

Let $\mathscr{L}$ be a very ample sheaf on a projective variety X. A coherent sheaf $\mathscr{F}$ on X is said to be normally presented if we have an exact sequence

$$R \otimes_k \mathscr{L}^{\otimes -1} \xrightarrow{\alpha} G \otimes_k \mathscr{O}_X \to \mathscr{F} \to 0$$

for some vector spaces R and G. Furthermore $\mathscr{F}$ is said to be strongly presented if the homomorphism $G \to \Gamma(X, \mathscr{F})$ is surjective.

Lemma 4. *A strongly presented coherent sheaf $\mathscr{F}$ is determined by $\Gamma(X, \mathscr{F})$ and the kernel of the multiplication*

$$\Gamma(X, \mathscr{F}) \otimes \Gamma(X, \mathscr{L}) \to \Gamma(X, \mathscr{F} \otimes \mathscr{L}).$$

Proof. First of all we may assume that $\beta\colon G \to \Gamma(X, \mathscr{F})$ is an isomorphism by factoring $G \otimes_k \mathscr{O}_X \to \mathscr{F}$ through $\overline{G} \otimes_k \mathscr{O}_X$ where $\overline{G}$ is the image of β. Then $R \to G \otimes \Gamma(X, \mathscr{L})$ has image in the kernel of multiplication. Hence the kernel contains enough relations to define $\mathscr{F}$ as a quotient sheaf of $G \otimes_k \mathscr{O}_X$. □

We will need a lemma to prove that some sheaves are strongly presented.

Lemma 5. *Given an exact sequence*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

of coherent sheaves on X*, assume that*

(a) $\mathscr{F}_2$ *is strongly normally presented, and*

(b) $\mathscr{F}_1$ *is generated by its sections and* $H^1(X, \mathscr{F}_1)$ *is zero. Then* $\mathscr{F}_3$ *is strongly normally presented.*

Proof. By (b) we have an exact sequence, $0 \to \Gamma(X, \mathscr{F}_1) \to \Gamma(X, \mathscr{F}_2) \to \Gamma(X, \mathscr{F}_3) \to 0$ and a surjection $\Gamma(X, \mathscr{F}_1) \otimes_k \mathscr{F}_X \to \mathscr{F}_1$. By (a) we have an exact sequence

$$R \otimes_k \mathscr{L}^{\otimes -1} \to \Gamma(X, \mathscr{F}_2) \otimes \mathscr{O}_X \to \mathscr{F}_2 \to 0$$

(using the proof of Lemma 4). Therefore we have an exact sequence

$$R \otimes_k \mathscr{L}^{\otimes -1} \oplus \Gamma(X, \mathscr{F}_1) \otimes_k \mathscr{O}_X \to \Gamma(X, \mathscr{F}_2) \otimes \mathscr{O}_X \to \mathscr{F}_3 \to 0$$

which we can factor as

$$R \otimes_k \mathscr{L}^{\otimes -1} \to \Gamma(X, \mathscr{F}_3) \otimes \mathscr{O}_X \to \mathscr{F}_3 \to 0. \quad \square$$

Remark. If we just assume that $\mathscr{F}_1$ is generated by its sections, then we can conclude that $\mathscr{F}_3$ is normally presented.

3. Abelian varieties

Let X be an abelian variety with ample invertible sheaf $\mathscr{L}$. An invertible sheaf $\mathscr{M}$ on X is said to be of type n if it is algebraically equivalent to $\mathscr{L}^{\otimes n}$ for some integer n. If the type is $\mathscr{M} \geq 1$ then $\mathscr{M}$ is ample and if it is ≥ 2 then $\mathscr{M}$ is generated by its sections and if it is ≥ 3 then $\mathscr{M}$ is very ample.

We have a basic result.

Theorem 6. *If* $\mathscr{N}$ *and* $\mathscr{M}$ *are two invertible sheaves on the abelian variety* X *such that* type$(\mathscr{N}) \geq 3$ *and* type$(\mathscr{M}) \geq 4$*, then* $\mathscr{N}$ *is strongly normally generated for* $\mathscr{M}$.

Proof. We first need to write enough relations between the sections of $\mathscr{N}$ and $\mathscr{M}$. Let Q_α be an invertible sheaf of type 2. We may write $\mathscr{N} = \mathscr{R}_\alpha \otimes Q_\alpha$ and $\mathscr{M} = \mathscr{S}_\alpha \otimes Q_\alpha$ where type$(\mathscr{R}_\alpha) \geq 1$ and type$(\mathscr{S}_\alpha) \geq 2$. Let $r \in \Gamma(X, \mathscr{R}_\alpha)$, $s \in \Gamma(X, \mathscr{S}_\alpha)$ and q_1 and $q_2 \in \Gamma(X, \mathscr{Q}_\alpha)$. Let $\langle\ ,\ \rangle$ denote the product of two sections. Evidently

$$a(r, s, q_1, q_2) = \langle r, q_1\rangle \otimes \langle s, q_w\rangle - \langle r, q_2\rangle \otimes \langle s, q_1\rangle$$

is contained in the kernel of the multiplication

$$\Gamma(X, \mathscr{N}) \otimes \Gamma(X, \mathscr{M}) \to \Gamma(X, \mathscr{N} \otimes \mathscr{M}).$$

Let A be the span of all possible such relations $a(r, s, q_1, q_2)$ for all possible α.

Let $N = \Gamma(X, \mathscr{N}) \otimes_k B/AB$ where B is the graded ring $\bigoplus_{n \geq 0} \Gamma(X, \mathscr{M}^{\otimes n})$. We have a canonical surjection $\pi: \widetilde{N} \to \mathscr{N}$ where $\widetilde{N}$ is the $\mathscr{O}_X$-module associated to the A-module N. The theorem will be proven if we can show that π is an isomorphism.

To do the above we must show that (1) for any point x of X the vector space $\widetilde{N}(x)$ is one-dimensional.

We may assume that all sheaves on X have been given compatible trivialization at x and let $e(\sigma) = \sigma(x)$ be evaluation. Then $\widetilde{N}(x) = \Gamma(X, \mathscr{N})/(1 \otimes e)A$ by definition. Thus we need to show that $(1 \otimes e)A$ has codimension one in $\Gamma(X, \mathscr{N})$. As the whole theorem is invariant under translation we may assume that x is the identity 0 of X.

Take $\lambda: \Gamma(X, \mathscr{N}) \to k$ a linear functional such that $\lambda((1 \otimes e)A) = 0$. Now $(1 \otimes e)a(r, s, q_1, q_1) = (1 \otimes s(0))(\langle r, q\rangle q_2(0) - \langle r, q_2\rangle q_1(0)$. As S_α is generated by its sections we may assume that $s(0) \neq 0$. Therefore $\lambda(\langle r, q_1\rangle q_2(0)) = \lambda(\langle r, q_2\rangle q_1(0))$ is symmetric in q_1 and q_2 and vanishes if $q_1(0)$ or $q_2(0)$ equals zero. Write $\lambda(\langle r, q_1\rangle q_2(0)) = \mu(r) q_1(0) q_2(0)$ and note that μ is well defined because Q_α is generated by its sections. We intend to show that (2) $\mu_\alpha(r) =$ constant $\cdot\, r(0)$. If we show (2) then λ is a multiple of evaluation at 0. Hence $(1 \otimes e)A$ is the kernel of evaluation and thus (1) is true.

We will show that (2) follows from a global variational argument with α. Let $\mathscr{P}$ be a Poincaré sheaf on $X \otimes \widehat{X}$ where $\widehat{X}$ is the dual abelian variety. Let $\mathscr{R}$ be one possible choice of $\mathscr{R}_\alpha$. Then all possible choices are the restriction of $\pi_X^* \mathscr{R} \otimes \mathscr{P}$ to the fibers of $\pi_{\widehat{X}}$. Globally μ_α is the value of a $\mathscr{O}_{\widehat{X}}$-homomorphism $\mu: \mathscr{W} \equiv \pi_{\widehat{X}*}(\pi_X^* \mathscr{R} \otimes \mathscr{P}) \to \mathscr{O}_X$. By [1,4] $H^{\dim \widehat{X}}(\widehat{X}, \mathscr{W})$ is onedimensional. Hence by duality $\mathrm{Hom}_{\widehat{X}}(\mathscr{W}, \mathscr{O}_{\widehat{X}})$ is one-dimensional but evaluation at 0 is one such homomorphism. Hence μ is a multiple of evaluation. Therefore (2) is true. □

When $\mathscr{N} = \mathscr{M}$ the theorem follows from D. Mumford's theorem [3, 4] that $\bigoplus_{k \geq 0} \Gamma(X, \mathscr{M}^{\otimes k})$ is almost normally presented as a ring. The proof of the theorem is close to Mumford's reasoning but the technicalities are easier.

4. The second method

An invertible sheaf $\mathscr{L}$ on the Jacobian J has type n if $\mathscr{L}$ is algebraically equivalent to $\mathscr{O}_J(n\theta)$ where θ is in the class of the principal polarization. Thus type$(\mathscr{L}_n) = n$ where $\mathscr{L}_n$ is the sheaf of §1, the notation of which we will be using.

Let $\mathscr{M}(\mathscr{R})$ be invertible sheaves on J of type $m(r)$. One might hope to prove that $\mathscr{U}_n(D) \otimes \mathscr{M}$ is normally presented for $\mathscr{R}$ for reasonable bounds on m, n and r. If one tries to use Lemma 2 and Lemma 5, the problem is that we would need $\mathscr{W}_{ng} \otimes \mathscr{M}$ to be generated by its sections (but I do not know when this is true). This emphasis is circumvented by applying the isogeny

$nL_J\colon J \to J$ given by multiplication by n. This resolves the problem. The result is

Theorem 7. (a) $((n1_J)^*\mathscr{U}_n(D)) \otimes \mathscr{M}$ *is normally presented for* $\mathscr{R}$ *if* $m \geq n + 2 \geq 4$ *and* $r \geq 4$.

(b) *It is strongly normally presented for* $\mathscr{R}$ *if* $m \geq n+2 \geq 4$, $r \geq 4$ *and, if* $n \geq 3$, *then* $g \geq 2$ *and* $m/(m,n)$ *prime to* $\operatorname{char}(k)$.

Proof. First of all we may assume that the effective divisor D consists of distinct point $e_1, \dots, e_d$. This follows because the isomorphism class of $\mathscr{U}_n(D)$ only depends on that of $\mathscr{L}_n|_C^{(D)} \equiv \mathscr{H}$ but we may vary $\mathscr{L}_n$ and D so that D is reduced while not changing $\mathscr{H}$.

Then from Lemma 2 we have an exact sequence

$$0 \to \mathscr{W}_{ng} \to \bigoplus_{1\leq i\leq d} \mathscr{S}_i \to \mathscr{U}_n(D) \to 0$$

where $\mathscr{S}_i = \pi_{J*}(Q_n|_{J\times e_i})$. Now $\operatorname{type}(\mathscr{S}_i) = 0$ because

$$Q_n|_{J\times e_i} = (\pi_1^*\mathscr{L}_n \otimes \mathscr{P})|_{J\times e_i} = \mathscr{L}_n(e_i) \otimes_k \mathscr{P}|_{J\times e_i},$$

which is algebraically equivalent to $\mathscr{O}_J$.

Next we pull this sequence back and get

(1) $0 \to (n1_J)^*\mathscr{W}_{ng} \to \bigoplus_{l\leq i\leq d} \mathscr{T}_i \to (n\times 1_J)^*\mathscr{U}_n(D) \to 0$, where $\operatorname{type}(\mathscr{T}_i) = 0$ and $\mathscr{T}_i = (n1_J)^*\mathscr{S}_i$. Now $(\bigoplus_{l\leq i\leq d} \mathscr{T}_i) \otimes \mathscr{M}$ is strongly normally presented for $\mathscr{R}$ if $m \geq 3$ and $r \geq 4$ by Theorem 6. Thus point (a) will follow from (1) $\otimes\, \mathscr{M}$ and the remark after Lemma 5 if we can prove that

(2) $(n1_J)^*\mathscr{W}_{ng} \otimes \mathscr{M}$ is generated by its sections if $m \geq n+2$. Also by Lemma 5 the point (b) will follow if we prove

(3) $H^1(J, (n1_J)^*\mathscr{W}_{ng} \otimes \mathscr{M}) = 0$ if $m \geq n+2 \geq 4$ and if $n \geq 3$ then $m > (n/(n-2))^{1/2g-1} n$ and $m/(m,n)$ prime to $\operatorname{char}(k)$.

To prove (2) we will use the surjection $\alpha\colon \mathscr{V}_n \twoheadrightarrow \mathscr{W}_{ng}$ of Proposition 1 as $n \geq 2$. Thus (2) will follow if we prove

(4) $(n1_J)^*\mathscr{V}_n \otimes \mathscr{M}$ is generated by its sections if $m \geq n+2$.

By [1] $(n1_J)^*\mathscr{V}_n \approx \Gamma(J, \mathscr{L}_n) \otimes_k \mathscr{L}_n^{\otimes -1}$. Hence we just need $(\mathscr{L}_n^{\otimes -1} \otimes \mathscr{R})$ to be generated by its sections; e.g. its type ≥ 2. As the type is $m-n$, (4) is true.

To prove (3) we have to modify the argument of [1] due to the presence of $(n1_J)^*$. We will first give some isomorphisms which follow in the same way as [1] from the vanishing of higher direct images and the Leray spectral sequence.

$$H^i(J(n1_J)^*\mathscr{W}_{ng} \otimes \mathscr{M} \simeq H^j(J \times C, \pi_C^*\mathscr{L}_n \otimes \mathscr{P}^n \otimes \pi_J^*\mathscr{M}|_{J\times C})$$

as

$$\begin{aligned}(n1_J)^*\mathscr{W}_{ng} &= \pi_{J*}((n1_J \times 1_C)^*(\pi_C^*\mathscr{L}_n \otimes \mathscr{P})|_{J\times C}) \\ &= \pi_{J*}(\pi_C^*\mathscr{L}_n \otimes \mathscr{P}^{\otimes n}|_{J\times C}).\end{aligned}$$

$H^i(J\times C, \pi_C^*\mathscr{L}_n \otimes \mathscr{P}^n \otimes \pi_J^*\mathscr{M}|_{J\times C)} = H^i(C, \mathscr{L}_n \otimes (n1_J)^*(\pi_{2*}(\mathscr{P}\otimes \pi_1^*\mathscr{M}))|C)$ as $\pi_{C*}\mathscr{L}_n|_C \otimes (\mathscr{P}^n \otimes \pi_J^*\mathscr{M})|_{J\times C}) \simeq \pi_{2*}(\pi_2^*\mathscr{L}_n \otimes (1_J \times n1_J)^*(\mathscr{P}\times \pi_1^*\mathscr{M}))|_C \simeq \mathscr{L}_n \otimes (n1_J)^*(\pi_{2*}(\mathscr{P}\otimes\pi_1^*\mathscr{M}))|_C$.

Thus

$$\circledast \qquad H^i(J(n1_J)^*\mathscr{W}_{ng}\otimes\mathscr{M}) \simeq H^i(C, \mathscr{L}_n \otimes (n1_J)^*(\pi_{2*}(\mathscr{P}\otimes\pi_1^*\mathscr{M}))|_C).$$

Now we need to determine when the cohomology of this sheaf on C vanishes. Let $a = m/(m,n)$. Write $m = ab$ and $n = cb$. As a is prime to the characteristic, the curve $C_a = (a1_J)^{-1}C$ is an unramified Galois covering of C. Hence for any quasicoherent sheaf $\mathscr{F}$ on C we have an injection $H^i(C\mathscr{F}) \hookrightarrow H^i(C_a, (a1_J)^*\mathscr{F})$. Thus we want to study

$$H^1(C_a, (a1_J)^*(\mathscr{L})n \otimes (n1_J)^*(\pi_{2*}(\mathscr{P}\otimes\pi_1^*\mathscr{M})))_{C_a}).$$

Here

$$\begin{aligned}&(a1_J)^*(\mathscr{L}_n \otimes (n1_J)^*(\pi_{2*}(\mathscr{P}\otimes\pi_1^*\mathscr{M}))) \\ &\quad\simeq (a1_J)^*\mathscr{L}_n \otimes (c1_J)^*(m1_J)^*(\pi_{2*}(\mathscr{P}\otimes\pi_1^*\mathscr{M})) \\ &\quad\simeq (a1_J)^*\mathscr{L}_n \otimes (c1_J)^*\mathscr{M}^{\otimes -1} \otimes_k \Gamma(J, \mathscr{M})\end{aligned}$$

as $(m1_J)^*(\pi_{2*}(\mathscr{P}\otimes\pi_1^*\mathscr{M})) \simeq \mathscr{M}^{\otimes -1} \otimes_k \Gamma(J,\mathscr{M})$. Thus the first cohomology groups vanish if

$$\begin{aligned}\deg((a1_J)^*\mathscr{L}_n \otimes (c1_J)^*\mathscr{M}^{\otimes -1}|_{C_a}) &> 2(\text{genus}(C_a) - 1) \\ &= 2\deg(a1_J)(g-1) \\ &= 2a^{2g}(g-1)\end{aligned}$$

but the degree of the sheaf is $a^{2g}ng - c^{2g}mg$. Finally for the vanishing we need the inequality $a^{2g}gn - c^{2g}gm > 2a^{2g}(g-1)$ or rather $2m^{2g-1}/gn + m^{2g-1} > n^{2g-1} + (2/n)m^{2g-1}$. For $n \geq 3$, this is true if $m > (n/(n-2))^{1/2g-1}n$, or, if $n = 2$ if $m > 2g^{1/2g-1}$. The simplest case of part (b) is $n = 3$ and $m = 5$ if $g \geq 2$ or $n = 2$ and $m = 4$. $\square$

The theorem tells us when $((nI_J)^*\mathscr{U}_n(D)) \otimes \mathscr{M}$ is determined by the multiplication

$$\beta\colon \Gamma(J, ((n1_J)^*\mathscr{U}_n(D))\otimes\mathscr{M}) \otimes \Gamma(J,\mathscr{R}) \to \Gamma(J, ((n1_J)^*\mathscr{U}_n(D))\otimes\mathscr{M}\otimes\mathscr{R}).$$

Thus we want to know more about this group. I will only give the dimension here.

Theorem 8. (a) $H^i(J,((n1_J)^*\mathscr{U}_n(D))\otimes\mathscr{M})=0$ *if* $i>0$ *and* $m\geq 1$.

(b) $\dim\Gamma(J,((n1_J)^*\mathscr{U}_n(D))\otimes\mathscr{M}) = ((d-gn+g-1)m^g+gn^2m^{g-1})$ *if* $m\geq 1$.

Proof. Consider the long exact sequence of cohomology of the sequence (1) tensored with $\mathscr{M}$. If $m\geq 1$ then the higher cohomology groups of the sheaf $\mathscr{T}_i\otimes\mathscr{M}$ vanish and by the isomorphism ⊛ of the last proof as

$$H^i(J,(n1_J)^*\mathscr{W}_{ng}\otimes\mathscr{M})=0 \quad \text{if } i\geq 1=\dim C.$$

Thus (a) is true by the long exact sequence.

By (a) the dimension is the Euler characteristic $\chi((n1_J)^*\mathscr{U}_n(D)\otimes\mathscr{M})$, which I intend to compute using the Hirzebruch-Riemann-Roch theorem. We need to know $\mathrm{ch}((n1_J)^*\mathscr{U}_n(D)\otimes\mathscr{M})$ because its number of codimension g cycles (= points) is the Euler characteristic as the Todd class of J is 1.

By Mattuck's result $c_t(\mathscr{U}_n(D))=\sum_{i\geq 0}w_it^i$ is algebraic equivalent where w_i is the image of $C^{(g-1)}$ in J. Thus by Poincaré relation, $c_t(\mathscr{U}_n(D))=\exp(\theta t)$ in numerical equivalence. Now if $c_t(\mathscr{U}_n(\theta))=\prod_{1\leq i\leq n}1+k_it$, we get $+t\theta = \log(\exp(\theta t)) = +\sum_i\log(1+w_it) = \sum_{p\geq 1}\sum_i(-1)^pw_i^{ptp}/p$ but $\mathrm{ch}_t(\mathscr{U}_n(D)) = \sum_i\exp(w_it)=\sum_p\sum_i w_i^p/p^i$. Comparing coefficients we find

$$\mathrm{ch}_t(\mathscr{U}_n(D))=\mathrm{rank}(\mathscr{U}_n(D)) + tc_1(\mathscr{U}_n(D))=\mathrm{rank}+t\theta,$$

where $\mathrm{rank}=-g\cdot n+d+g=1$. Hence $\mathrm{ch}((n1_J)^*\mathscr{U}_n(D))=\mathrm{rank}+n^2\theta$. Thus

$$\mathrm{ch}((n1_J)^*\mathscr{U}_n(D)\otimes\mathscr{M}=\mathrm{ch}((n1_J)^*\mathscr{U}_n(D))\cdot\mathrm{ch}(\mathscr{M})=\mathrm{rank}+n^2\theta)\exp(m\theta).$$

Therefore $\chi((n1_J)^*\mathscr{U}_n(D)\otimes\mathscr{M}) = (\mathrm{rank})m^g+n^2m^{g-1}g$ and the result follows. □

A last remark is

Theorem 9. *In the range of* Theorem 7(b) *then the multiplication* β *is surjective.*

Proof. The conditions of Theorem 7(b) are true when $\mathscr{M}=\mathscr{M}\otimes\mathscr{R}$. The proof shows that the homomorphism

$$\Gamma(J,\oplus\mathscr{T}_i\otimes\mathscr{M})\to\Gamma(J,(m1_J)^*\mathscr{U}_n(D)\otimes\mathscr{M})$$

is surjective for $\mathscr{M}=\mathscr{M}$ and $\mathscr{M}\otimes\mathscr{R}$. This theorem results because the multiplication

$$\Gamma(J,\mathscr{T}_i\otimes\mathscr{M})\otimes\Gamma(J,\mathscr{R})\to\Gamma(J,\mathscr{T}_i\otimes\mathscr{M}\otimes\mathscr{R})$$

is surjective by Mumford's result in [4]. □

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