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MINIMAL HARMONIC FUNCTIONS ON DENJOY DOMAINS

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ABSTRACT. Let $\Omega = \mathbb{R}^n \backslash E$, where E is a closed subset of the hyperplane $\{x_n = 0\}$ and every point of E is regular for the Dirichlet problem on Ω. Further, let α_k denote the $(n-1)$-dimensional measure of the set $\{X \in \Omega: x_n = 0, e^k < |X| < e^{k+1}\}$. It is known that the cone, $\mathscr{P}_E$, of positive harmonic functions on Ω which vanish on E has dimension 1 or 2. In this paper it is shown that if $\sum e^{-nk}\alpha_k^{n/(n-1)} < +\infty$, then $\dim \mathscr{P}_E = 2$. This result, which in the case $n = 2$ implies a recent theorem of Segawa, is also shown to be sharp.

1. INTRODUCTION AND RESULTS

Points of $\mathbb{R}^n$ $(n \geq 2)$ are denoted by $X = (X', x_n)$, where $X' \in \mathbb{R}^{n-1}$. We call Ω a *Denjoy domain* if $\Omega = \mathbb{R}^n \backslash E$, where E is a nonempty closed proper subset of the hyperplane $\{x_n = 0\}$ such that each point of E is regular for the Dirichlet problem in Ω.

Let $\mathscr{P}_E$ denote the cone of positive harmonic functions on Ω which vanish on E. It is known (see [1] or [2]) that either all functions in $\mathscr{P}_E$ are proportional or $\mathscr{P}_E$ is generated by two linearly independent minimal harmonic functions. (A positive harmonic function u on Ω is called *minimal* if any other positive harmonic function v on Ω satisfying $v \leq u$ is proportional to u.) We describe these cases by writing $\dim \mathscr{P}_E = 1$ and $\dim \mathscr{P}_E = 2$ respectively.

Roughly speaking, $\dim \mathscr{P}_E = 2$ if the set $\{x_n = 0\}\backslash E$ is "sufficiently sparse near infinity". Several results in this direction can be found in Benedicks [2]. Recently Segawa [4], working in the complex plane, added the following.

Theorem A. *Let $n = 2$. If there exists $\lambda > \frac{1}{2}$ such that*

$$\int_{\{x:\ (x,0)\in\Omega,\ |x|\geq t\}} x^{-2}\,dx = O(t^{-1}[\log t]^{-\lambda}) \qquad (t \to +\infty), \tag{1}$$

then $\dim \mathscr{P}_E = 2$.

(In fact, Segawa stated his result for harmonic functions with pole at the origin rather than at infinity. The above is an equivalent formulation based on inversion in the unit circle.)

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The purpose of this paper is to provide a result of this type for $\mathbb{R}^n$, which in the case $n = 2$ improves Theorem A. We will also show that our condition is sharp.

For each $k \in \mathbb{N}$ let

$$F_k = \{X' \in \mathbb{R}^{n-1} : (X', 0) \in \Omega \text{ and } e^k < |X'| < e^{k+1}\},$$

and put $\alpha_k = |F_k|$, the $(n-1)$-dimensional Lebesgue measure of F_k. Our main result is as follows.

Theorem 1. *If* $\sum e^{-nk}\alpha_k^{n/(n-1)} < +\infty$, *then* $\dim \mathscr{P}_E = 2$.

The proof of Theorem 1 relies on the work of Benedicks mentioned above and a balayage-type argument. Details can be found in §2. To see that Theorem 1 includes Theorem A, note that if $n = 2$ and (1) hold, then

$$e^{-k}\alpha_k \leq e^{2+k}\int_{F_k} x^{-2}\,dx = O(k^{-\lambda}) \qquad (k \to \infty),$$

so $\sum e^{-2k}\alpha_k^2 < +\infty$.

In the case $n \geq 3$ it is known [2, Corollary 1] that if E omits an infinite $(n-1)$-dimensional circular cone, then $\dim \mathscr{P}_E = 1$. The following result shows that it is sufficient for E to omit a certain sequence of $(n-1)$-dimensional balls. It also establishes the sharpness of Theorem 1.

Theorem 2. *Let* (r_k) *be a sequence of nonnegative real numbers such that* $\sum e^{-nk} r_k^n = +\infty$. *If*

$$E = \{x_n = 0\} \setminus \bigcup_{k=1}^{\infty} \{(X', 0) : |X' - (2e^k, 0, \ldots, 0)| < r_k\},$$

then $\dim \mathscr{P}_E = 1$.

The proof of Theorem 2 is given in §3.

2. Proof of Theorem 1

2.1. We begin by stating a result of Benedicks on which our proofs rely. For each $X' \in \mathbb{R}^{n-1}$ let $K(X')$ be the open cube in $\mathbb{R}^n$ with center $(X', 0)$ and side $e^{-1}|X'|$, all sides being parallel to the coordinate hyperplanes. Further, let $\Omega(X') = K(X') \setminus E$ and $\beta_E(X')$ denote the harmonic measure of $\partial K(X')$ on $\Omega(X')$ evaluated at the point $(X', 0)$. The result below is drawn from [2, Theorems 3 and 4].

Theorem B. *The following are equivalent*:

(i) $\dim \mathscr{P}_E = 2$;

(ii) *there exists a function* $u \in \mathscr{P}_E$ *such that* $u(X) \geq |x_n|$ *on* $\mathbb{R}^n$;

(2) (iii) $\displaystyle\int_{\{|X'| \geq 1\}} |X'|^{1-n} \beta_E(X')\,dX' < +\infty$.

2.2. We now collect together some definitions required in the proof of Theorem 1.

Let h denote the function given by $h(X) = |x_n|$, and let σ_n be the surface area of the unit sphere in $\mathbb{R}^n$. Also, let

$$F_0 = \{X' \in \mathbb{R}^{n-1} : (X', 0) \in \Omega \text{ and } |X'| < e\}$$

and $\alpha_0 = |F_0|$. For each $k \in \mathbb{N}$, we let

$$\gamma_k = \sum_{j=1}^{4} \alpha_{4k-j}$$

and note that the hypothesis of Theorem 1 implies that the series $\sum e^{-4nk} \gamma_k^{n/(n-1)}$ converges. We define the open sets

$$G_k = \left[\bigcup_{j=1}^{4} F_{4k-j}\right] \times (-\gamma_k^{1/(n-1)}, \gamma_k^{1/(n-1)}) \text{ and } U_k = \bigcup_{j=1}^{k} G_j .$$

(If $\gamma_k = 0$, then G_k is empty.)

If f is a function defined on the boundary ∂W of a bounded open set W, we use $H[W, f]$ to denote the Perron-Wiener-Brelot solution (if it exists) to the corresponding Dirichlet problem on W. (An account of the properties of $H[W, f]$ can be found in [3, Chapter 8].) If f is defined on the hyperplane $\{x_n = 0\}$, we similarly write $I[f]$ for the corresponding half-space Poisson integral; that is,

$$I[f](X) = 2x_n \sigma_n^{-1} \int_{\mathbb{R}^{n-1}} \frac{f(Y', 0)\, dY'}{\{|X' - Y'|^2 + x_n^2\}^{n/2}} \qquad (x_n > 0).$$

Now suppose that s is a nonnegative subharmonic function on $\mathbb{R}^n$. We define

$$(J_0 s)(X) = \begin{cases} h(X) + I[s](X', |x_n|) & (x_n \neq 0) \\ s(X) & (x_n = 0) \end{cases}$$

and

$$(J_k s)(X) = \begin{cases} H[U_k, s](X) & (X \in U_k) \\ s(X) & (X \in \mathbb{R}^n \backslash U_k) \end{cases}$$

for any $k \in \mathbb{N}$.

2.3. For any set $A \subseteq \mathbb{R}^{n-1}$, let χ_A denote the function valued 1 on $A \times \{0\}$ and 0 elsewhere on $\{x_n = 0\}$. Also, let r_{n-1} be the radius of the ball B centered at the origin O' of $\mathbb{R}^{n-1}$ for which $|B| = 1$. We will need the following simple lemma.

Lemma 1. *There is a positive constant $c_n < 1$, depending only on n, such that if A is a measurable subset of $\mathbb{R}^{n-1}$ satisfying $|A| = 1$, then*

$$I[\chi_A](X', 1) \le c_n \qquad (X' \in \mathbb{R}^{n-1}).$$

In fact,

$$\begin{aligned}
I[\chi_A](X',1) &= \frac{2}{\sigma_n}\int_A \frac{dY'}{\{|X'-Y'|^2+1\}^{n/2}} \\
&\le \frac{2}{\sigma_n}\int_{\{Y'\in A:\,|X'-Y'|\le r_{n-1}\}} \frac{dY'}{\{|X'-Y'|^2+1\}^{n/2}} \\
&\quad + \frac{2}{\sigma_n}(r_{n-1}^2+1)^{-n/2}|\{Y'\in A:\,|X'-Y'|>r_{n-1}\}| \\
&\le \frac{2}{\sigma_n}\int_{\{Y':\,|X'-Y'|\le r_{n-1}\}} \frac{dY'}{\{|X'-Y'|^2+1\}^{n/2}} \\
&= I[\chi_B](O',1).
\end{aligned}$$

2.4. We claim that it is enough to prove Theorem 1 in the special case where $\alpha_{4k}=0$ for all $k=0,1,2,\ldots$. To see this, let

$$D=\bigcup_{k=1}^{\infty}\{(X',0):\, e^{4k}\le |X'|\le e^{4k+1}\},$$

$$E_j = D\cup\{(e^jX',0):\,(X',0)\in E\}\qquad (j=0,1,2,3),$$

and observe that

$$\beta_{E_j}(X')=\beta_E(e^{-j}X')\qquad (e^{4k+2}<|X'|<e^{4k+3};\, k=0,1,2,\ldots). \tag{3}$$

Now assume that the special case of the theorem has been established. Then we know that $\dim\mathscr{P}_{E_j}=2$ for $j=0,1,2,3$; and so by Theorem B, (2) holds, with E replaced by any E_j. From (3) it follows that

$$\int_{\bigcup_k\{X':\, e^{4k+j}<|X'|<e^{4k+j+1}\}} |X'|^{1-n}\beta_E(X')\,dX' < +\infty$$

for each j. Hence (2) holds, and by a further application of Theorem B, $\dim\mathscr{P}_E=2$.

In the proof of Theorem 1 we can also assume that the sum T of the series $\sum e^{-4nk}\gamma_k^{n/(n-1)}$ satisfies

$$c_n+2T\sigma_n^{-1}(e^{-3}-e^{-7/2})^{-n}<1, \tag{4}$$

for the convergence of (2) is unaffected when we replace E by $E\cup\{(X',0):\,|X'|\le a\}$ for any $a>0$. The left-hand side of (4) will be denoted by d_n.

2.5. In the light of §2.4 it remains to prove Theorem 1 under the additional assumptions that (4) holds and that $\alpha_{4k}=0$ for all $k=0,1,2,\ldots$.

Let $k\in\mathbb{N}$ and suppose that s is a nonnegative subharmonic function on $\mathbb{R}^n$ which vanishes on $E\cup\{(X',0):\,|X'|\ge e^{4k-4}\}$. Suppose also that the δ-subharmonic function $s-h$ is bounded above on $\mathbb{R}^n$. Then the function $S=J_ks$ is also nonnegative and subharmonic on $\mathbb{R}^n$ (the regularity of U_k for the Dirichlet problem follows from the regularity of Ω); we have $S\ge s$; the function $S - h$ is bounded above; and S vanishes on $E \cup \{(X', 0): |X'| \geq e^{4k}\}$. A Phragmén-Lindelöf argument applied to each half-space now shows that $J_0 S \geq S$ on $\{X: x_n \neq 0\}$ and thus that $J_0 S$ is subharmonic on $\mathbb{R}^n$. Again $J_0 S - h$ is bounded above and $J_0 S$ vanishes on $E \cup \{(X', 0): |X'| \geq e^{4k}\}$.

It follows that the functions (s_k) defined inductively by

$$s_0 = h; \qquad s_{2k-1} = J_k s_{2k-2}; \qquad s_{2k} = J_0 s_{2k-1}$$

are all nonnegative and subharmonic on $\mathbb{R}^n$, vanish on E, and form an increasing sequence.

2.6. To prove that $\lim_{k\to\infty} s_k$ is finite, we establish the following lemma.

Lemma 2. *Let $k \in \mathbb{N}$. If there is a positive constant C such that*

$$s_{2k-1}(X', 0) \leq C\gamma_j^{1/(n-1)} \qquad ((X', 0) \in G_j; j \in \mathbb{N}), \tag{5}$$

then

$$s_{2k+1}(X', 0) \leq (1 + Cd_n)\gamma_j^{1/(n-1)} \qquad ((X', 0) \in G_j; j \in \mathbb{N}).$$

To see this, suppose (5) holds, let $j \in \mathbb{N}$ be such that $\gamma_j \neq 0$, and let

$$K = \{X: e^{(8j-7)/2} < |X'| < e^{(8j+1)/2} \text{ and } |x_n| < \gamma_j^{1/(n-1)}\}.$$

Using the fact that $\alpha_{4j-4} = 0 = \alpha_{4j}$, we have, for any $X \in \partial K$,

$$\begin{aligned}
&2x_n \sigma_n^{-1} \int_{\{Y' \in \mathbb{R}^{n-1}:\, (Y',0) \notin K\}} \frac{s_{2k-1}(Y', 0)\, dY'}{\{|X' - Y'|^2 + x_n^2\}^{n/2}} \\
&\quad \leq 2\sigma_n^{-1} \gamma_j^{1/(n-1)} C \left\{ [e^{(8j-7)/2} - e^{4j-4}]^{-n} \sum_{i=1}^{j-1} \gamma_i^{n/(n-1)} \right. \\
&\qquad\qquad \left. + \sum_{i=j+1}^{k} [e^{4i-3} - e^{(8j+1)/2}]^{-n} \gamma_i^{n/(n-1)} \right\} \\
&\quad \leq 2\sigma_n^{-1} \gamma_j^{1/(n-1)} C \left\{ [e^{1/2} - 1]^{-n} \sum_{i=1}^{j-1} e^{-4ni} \gamma_i^{n/(n-1)} \right. \\
&\qquad\qquad \left. + [e^{-3} - e^{-7/2}]^{-n} \sum_{i=j+1}^{k} e^{-4ni} \gamma_i^{n/(n-1)} \right\} \\
&\quad \leq 2\sigma_n^{-1} \gamma_j^{1/(n-1)} C [e^{-3} - e^{-7/2}]^{-n} T,
\end{aligned}$$

where T is as defined in §2.4. Also, using Lemma 1 and the dilation $X \mapsto \gamma_j^{1/(n-1)} X$, we have

$$\begin{aligned}
&2x_n \sigma_n^{-1} \int_{\{Y' \in \mathbb{R}^{n-1}:\, (Y',0) \in K\}} \frac{s_{2k-1}(Y', 0)\, dY'}{\{|X' - Y'|^2 + x_n^2\}^{n/2}} \\
&\quad \leq \gamma_j^{1/(n-1)} C \max\{c_n, 2\sigma_n^{-1}[e^{-3} - e^{-7/2}]^{-n} e^{-4nj} \gamma_j^{n/(n-1)}\} \\
&\quad = \gamma_j^{1/(n-1)} C c_n \qquad (X \in \partial K).
\end{aligned}$$

(The last step is possible since, by the final paragraph of §2.4, we can assume j to be sufficiently large to achieve this.) Hence, if $X \in \partial K$,

$$\begin{aligned} s_{2k}(X) &= (J_0 s_{2k-1})(X) \\ &\leq \gamma_j^{1/(n-1)}\{1 + C[2T\sigma_n^{-1}(e^{-3} - e^{-7/2})^{-n} + c_n]\} \\ &= \gamma_j^{1/(n-1)}(1 + Cd_n), \end{aligned}$$

where d_n is as defined in §2.4. It follows that, if $(X', 0) \in G_j$, then

$$\begin{aligned} s_{2k+1}(X', 0) &= (J_{k+1} s_{2k})(X', 0) \\ &= \begin{cases} H[G_j, s_{2k}](X', 0) & (j \leq k+1) \\ 0 & (j > k+1) \end{cases} \\ &\leq \max\{s_{2k}(Y) \colon Y \in \partial K\} \\ &\leq \gamma_j^{1/(n-1)}(1 + Cd_n), \end{aligned}$$

the penultimate step being a consequence of the maximum principle and the fact that $G_j \subseteq K$. The lemma is now established.

2.7. We now complete the proof of Theorem 1. Since

$$s_1(X', 0) = (J_1 h)(X', 0) \begin{cases} \leq \gamma_1^{1/(n-1)} & ((X', 0) \in G_1) \\ = 0 & ((X', 0) \notin G_1) \end{cases}$$

it follows from Lemma 2 that

$$s_{2k+1}(X', 0) \leq (1 + d_n + d_n^2 + \cdots + d_n^k)\gamma_j^{1/(n-1)} \qquad ((X', 0) \in G_j; j \in \mathbb{N}).$$

Since $d_n < 1$, it follows that the function $s = \lim_{k\to\infty} s_k$ is locally bounded on $\{x_n = 0\}$. Further, s_{2k-1} is harmonic in U_k, so $s = \lim s_{2k-1}$ is harmonic in $\bigcup_j G_j$. We can assume this latter set to be nonempty (otherwise the theorem is vacuously true), so there are points of the half-spaces $\{x_n > 0\}$ and $\{x_n < 0\}$ where s is finite. Since s_{2k} is harmonic in $\{x_n \neq 0\}$ for each k, so is $s = \lim s_{2k}$. In fact, applying the monotone convergence theorem to the equation $s_{2k} = J_0 s_{2k-1}$, we find that

$$s(X) = h(X) + I[s](X', |x_n|) \qquad (x_n \neq 0),$$

and so s is locally bounded on $\mathbb{R}^n$.

We have now established that s is harmonic on the set $\mathbb{R}^n \backslash (E \cup L)$, where

$$L = \{(X', 0) \colon \log|X'| \in \mathbb{N}\}.$$

Since L is polar and s is locally bounded on $\mathbb{R}^n$, it follows that s can be redefined (if necessary) on $L \cap E$ in such a way that s is harmonic on $\mathbb{R}^n \backslash E$.

Now let $a > 0$ and $B = \{X \in \Omega \colon |X| < a\}$. Since each s_k is subharmonic on $\mathbb{R}^n$ we have $s_k \leq H[B, s_k]$ on B, whence $s \leq H[B, s]$ on B by monotone convergence. Since, further, each s_k vanishes on E, it follows that s continuously vanishes on the set $\{X \in E \colon |X| < a\}$. (Any point of the latter set is regular for the Dirichlet problem on B.) Thus, as a can be arbitrarily large, s continuously vanishes on E. Also, $s \geq s_0 = h$. The equivalence of (i) and (ii) in Theorem B now shows that $\dim \mathscr{P}_E = 2$.

3. Proof of Theorem 2

Let $E' = \{(X', 0) : |X'| \geq 1\}$ and let $u \in \mathscr{P}_{E'}$ be symmetric in x_n. Clearly there is a positive constant c such that

$$u(X) = c|x_n| + I[u](X', |x_n|) \qquad (x_n \neq 0)$$

and so, multiplying by a suitable factor, we can assume that $u(X) \leq 1 + |x_n|$ for all $X \in \mathbb{R}^n$.

Let (r_k) and E be as in the statement of Theorem 2 and let $2\rho_k = \min\{r_k, e^k\}$. Clearly $\sum e^{-nk}\rho_k^n = +\infty$. Let

$$B_k = \{X' \in \mathbb{R}^{n-1} : |X' - (2e^k, 0, \dots, 0)| < \rho_k\}.$$

If $X' \in B_k$, then $\beta_E(X') \geq \beta_*(X')$, where $\beta_*(X')$ is the harmonic measure of $\partial K(X')$ on the set

$$W = K(X') \backslash \{(Y', 0) : |Y' - X'| \geq \rho_k\}$$

evaluated at the point $(X', 0)$. Applying the maximum principle to W, it follows that

$$\beta_*(X') \geq \left(1 + \frac{|X'|}{2e\rho_k}\right)^{-1} u(0),$$

whenever $\rho_k \neq 0$. Hence

$$\begin{aligned}\int_{\{|X'|\geq 1\}} |X'|^{1-n}\beta_E(X')\,dX' &\geq u(0)\sum_{\rho_k\neq 0}\int_{B_k}|X'|^{1-n}\left(1 + \frac{|X'|}{2e\rho_k}\right)^{-1} dX' \\ &\geq \frac{\sigma_{n-1}}{(n-1)}u(0)\sum_{\rho_k \neq 0} e^{(k+1)(1-n)}\rho_k^{n-1}\left(1 + \frac{e^k}{2\rho_k}\right)^{-1} \\ &\geq \frac{\sigma_{n-1}}{(n-1)}u(0)e^{1-n}\sum_{k=1}^{\infty} e^{-nk}\rho_k^n \\ &= +\infty.\end{aligned}$$

It now follows from Theorem B that $\dim \mathscr{P}_E = 1$.

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