GAUSSIAN CURVATURES OF LORENTZIAN METRICS
ON THE PLANE AND PUNCTURED PLANES

JIANGFAN LI

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Dedicated to Professors Gu Chaohao and Hu Hesheng

Abstract. We prove that every $f \in C^k(\mathbb{R}^2)$ is the Gaussian curvature of some $C^{k+1}$-Lorentzian metric $(0 \leq k \leq \infty)$. Let $M$ denote the cylinder. We prove that every continuous function on $M$ is the Gaussian curvature of some $C^1$-Lorentzian metric. If $f \in C^k(M)$ satisfies the condition (H) in the Lemma 2 below, then it is the curvature function of some $C^{k+1}$-Lorentzian metric. If $f \in C^k(\mathbb{R}^2)$ $(1 \leq k \leq \infty)$ has compact support, then the Lorentzian metric can be made complete.

1. Introduction

Given a function on the surface, does there exist a metric whose Gaussian curvature is the given function? This is to solve the second order nonlinear partial differential equation:

$$K(l) = f,$$

where $f$ is the given function, $K(l)$ is the Gaussian curvature of the definite or indefinite metric $l$. Many mathematicians have studied the case when $l$ is a Riemannian metric. For the Lorentzian metric $l$, Burns [1] got some results in 1977. In this note we are going to solve (1) on the 2-dimensional plane and to make a try on punctured planes (skirts, shirts, T-shirts, etc.).

2. Curvature functions on the plane

Lemma 1. Suppose $h \in L^1(\mathbb{R}^2), \|h\|_{L^1(\mathbb{R}^2)} < 1/\pi$; then

$$w(x, y) = \int_0^x \int_0^y h(s, t)e^{w(s, \xi)} ds \, dt \quad \text{on } \mathbb{R}^2$$

admits a unique solution $w \in C(\mathbb{R}^2)$.
Proof. Define a sequence by
\[ u_0 = 0, \]
\[ u_{n+1}(x, y) = \int_0^x \int_0^y h(s, t) e^{u_n(s, t)} ds \, dt, \quad n = 0, 1, 2, \ldots; \]
then \( \|u_n\|_{C(R^2)} < 1, \|u_{n+1} - u_n\|_{C(R^2)} < (\varepsilon/\pi)\|u_n - u_{n-1}\|_{C(R^2)}. \) Hence \( \{u_n\} \)
converges uniformly on \( R^2 \) and the limit function is a solution of the equation (2). The uniqueness is obvious. Q.E.D.

**Theorem 1.** Let \( k \) be a nonnegative integer or \( k = \infty \). Suppose \( f \in C^k(R^2) \). Then there exists a \( C^{k+1} \)-Lorentzian metric on \( R^2 \) which is pointwise conformal to the standard flat Lorentzian metric \( dx \, dy \) such that its Gaussian curvature equals \( f \).

**Proof.** Let \( G \in C^\infty(R) \) such that, for all \( t \in R \)
\[ G(t) > \max\{\pi(|f(x, y)| + 1) \mid |x| \leq |y| = |t| \text{ or } |y| \leq |x| = |t|\}. \]
Let
\[ h(x, y) = -\frac{1}{2} f(x, y) e^{-x^2 - y^2}; \]
then
\[ \|h\|_{L^1(R^2)} < \frac{1}{\pi}. \]
By Lemma 1, there exists \( w \in C(R^2) \) satisfying the equation (2). Then
\[ u(x, y) = w(x, y) - x^2 - y^2 - \ln(G(x)G(y)) \]
satisfies
\[ u_{xy} = -\frac{1}{2} f(x, y) e^u \quad \text{on } R^2, \]
which means the metric \( e^u dx \, dy \) has the Gaussian curvature \( f \). Q.E.D.

As a consequence, given a \( C^k \)-function \( f \) on any 2-manifold \( M \), for each point \( p \in M \), locally there always exists a \( C^{k+1} \)-Lorentzian metric whose Gaussian curvature equals \( f \) in a neighborhood of \( p \).

### 3. Curvature functions on a punctured plane

**Lemma 2.** Let \( k \) be a nonnegative integer or \( k = \infty \), \( p = (x_0, y_0) \in R^2 \). Suppose \( f \in C^k(R^2 \setminus \{p\}) \) satisfies the following condition:

For each \( n \) \((0 \leq n < k + 1)\), there exists a function \( F \in L^1[-1, 1] \) \( \{F(0) = +\infty\}, \) such that, for \( (x, y) \in [-1, 1]^2 \setminus \{(0, 0)\} \),
\[ \left| \frac{\partial}{\partial x_n} f(x + x_0, y + y_0) \right| + \left| \frac{\partial}{\partial y_n} f(x + x_0, y + y_0) \right| < \min\{F(x), F(y)\}. \]
Then any solution of the equation (2) (replacing \( h \) by \( f \)) is in fact in \( C^{k+1}(\mathbb{R}^2 \setminus \{p\}) \).

**Proof.** By Lebesgue’s dominated convergence theorem. Q.E.D.

**Theorem 2.** Let \( k \) be a nonnegative integer or \( k = \infty \). \( p = (0,0) \in \mathbb{R}^2 \). Suppose \( f \in C^k(\mathbb{R}^2 \setminus \{p\}) \) satisfies the condition (H) of Lemma 2; then there exists a \( C^{k+1} \)-Lorentzian metric on \( \mathbb{R}^2 \setminus \{p\} \) such that its Gaussian curvature equals \( f \).

**Proof.** There is \( G \in C^\infty(\mathbb{R}) \) satisfying (3) for \( |t| > 1 \). Then

\[
g(x,y) = -\frac{f(x,y)}{2G(x)G(y)}e^{-x^2-y^2}
\]

is integrable on \( \mathbb{R}^2 \). Pick \( \epsilon > 0 \) such that

\[
h = \epsilon g
\]

satisfies \( \|h\|_{L^1(\mathbb{R}^2)} < 1/\pi \). Clearly \( h \) also satisfies the condition (H) of Lemma 2. Hence the solution \( w \) to the equation (2) (in Lemma 1) is in \( C^{k+1}(\mathbb{R}^2 \setminus \{p\}) \) by Lemma 2, and so is the function \( u \) which is defined by

\[
u(x,y) = w(x,y) - x - y - \ln(G(x)G(y)) + \ln\epsilon.
\]

Then the Lorentzian metric \( e^u dx\,dy \), which has the Gaussian curvature \( f \), is of class \( C^{k+1} \). Q.E.D.

**Remark 1.** Let \( M = \mathbb{R}^2 \setminus \{ \text{finite points} \} \), \( f \in C^k(M) \). If for each \((x_0,y_0) \in \mathbb{R}^2 \setminus M\), \( f \) satisfies the condition (H) of Lemma 2, then the same conclusion holds.

**Theorem 3.** On a cylinder, every continuous function is the Gaussian curvature for some \( C^1 \)-Lorentzian metric.

**Proof.** Look at a cylinder as \( \mathbb{R}^2 \setminus \{(0,0)\} \), say, \( p = (0,0) \). Suppose \( f \in C(\mathbb{R}^2 \setminus \{p\}) \).

We first shrink \( f \) into control. Pick \( g \in C(\mathbb{R}^2 \setminus \{(0,0)\}) \), such that

\[
g(x,y) = h(r) \text{ depends only on } r = \sqrt{x^2 + y^2},
\]

\[
g > f \text{ on } \mathbb{R}^2 \setminus \{(0,0)\},
\]

\[
h(r) \text{ is decreasing for } r \in (0,2].
\]

Let \( \varphi \) be a \( C^\infty \)-diffeomorphism of \((0, + \infty)\) such that \( \varphi|_{(1, + \infty)} = \text{id} \) and \( h(\varphi) \in L^1(0, 1] \). Define

\[
\Phi: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\},
\]

\[
\Phi(x,y) = \left( \frac{\varphi(r)}{r}x, \frac{\varphi(r)}{r}y \right),
\]

where \( r = \sqrt{x^2 + y^2} \). Define

\[
F(t) = h(\varphi(||t||)) \quad \text{for } t \neq 0, \quad \text{and } F(0) = +\infty.
\]
Then $F \in L^1[-1, 1]$ and for $(x, y) \in [-1, 1] \setminus \{(0, 0)\}$,

$$f(\Phi(x, y)) = f \left( \frac{\varphi(r)x}{r}, \frac{\varphi(r)y}{r} \right) < g \left( \frac{\varphi(r)x}{r}, \frac{\varphi(r)y}{r} \right)$$

$$= h(\varphi(r)) = F(r) \leq \min(F(x), F(y)).$$

Applying Theorem 2 we know that $f(\Phi)$ is the curvature function of some $C^1$-Lorentzian metric $l$. Hence the pull-back metric $(\Phi^{-1})^*l$ has the curvature $f$. Q.E.D.

Remark 2. The same conclusion holds for the manifold $R^2 \setminus A$ where $A$ is a discrete subset of $R^2$.

4. CURVATURE FUNCTIONS FOR COMPLETE LORENTZIAN METRICS

Throughout this section, we use the notation $B_r = \{x^2 + y^2 \leq r^2\}$. For the Lorentzian metric $e^{u}dx\,dy$ on $R^2$, the equations of the geodesic are

$$\ddot{x} + u_x\dot{x}^2 = 0,$$

$$\ddot{y} + u_y\dot{y}^2 = 0.$$ 

In particular, the characteristic lines (i.e. those lines parallel to the $x$-axis or $y$-axis) are geodesics. We say a geodesic $\gamma(t)$ is forward complete (complete, respectively) if it exists for all $t \geq 0$ (all $t$, resp.). Note the fact that for a positive function $f \in C(R)$, the solution to the problem

$$\dot{x} = f(x), \quad x(0) = x_0$$

exists for all $t \geq 0$ if and only if

$$\int_{x_0}^{+\infty} \frac{dx}{f(x)} = +\infty.$$

This implies

Lemma 3. Let $u \in C(R^2)$. The characteristic lines of the metric $e^{u}dx\,dy$ are complete if and only if for any $x$ and $y$,

$$\int_{0}^{+\infty} e^{u(x,y)}\,dx = \int_{0}^{+\infty} e^{u(x,y)}\,dy = \int_{-\infty}^{0} e^{u(x,y)}\,dx = \int_{-\infty}^{0} e^{u(x,y)}\,dy = +\infty.\quad (4)$$

Proof. Consider the characteristic line $\gamma(t) = (x(t), y_0)$. It is a geodesic if properly parametrized:

$$\ddot{x}(t) = -u_x(x(t), y_0)\dot{x}(t)^2, \quad x(0) = x_0, \quad \dot{x}(0) \neq 0.$$ 

Let

$$p(t) = \dot{x}(t);$$

then

$$\frac{dp}{dx} = -u_x(x, y_0)p.$$ 

Therefore

$$p = c e^{u(x, y_0)} \quad (c = \dot{x}(0)e^{u(x_0, y_0)} \neq 0).$$
That is
\[ \frac{dx}{dt} = ce^{-u(x,y_0)}. \]

Hence \( \gamma(t) \) is forward complete if and only if \( x(t) \) is, if and only if
\[ \int_{x_0}^{\infty} e^{u(x,y_0)} dx = +\infty \quad (\text{if } c > 0) \]
or
\[ \int_{-\infty}^{x_0} e^{u(x,y_0)} dx = +\infty \quad (\text{if } c < 0). \]

All characteristic lines are complete if and only if every characteristic line is forward complete, if and only if (4) holds. Q.E.D.

Remark 3. By Lemma 3 one can see that the metric constructed in the proof of Theorem 1 is not complete.

Corollary 1. If \( u \in C(R^2) \) satisfies
\[ \liminf_{r \to \infty} \left( \|u\|_{C(B_r)} - \ln R \right) < +\infty, \]
then every characteristic line of the Lorentzian metric \( e^u dx dy \) on \( R^2 \) is complete.

Proof. It is equivalent to prove that the integrals in Lemma 3 are \( +\infty \). From the condition, there exist a sequence \( r_n \to +\infty \) and a constant \( c > 0 \) such that
\[ \|u\|_{C(B_{r_n})} - \ln r_n \leq \ln c. \]

Then
\[ e^u \geq \frac{1}{cr_n} \quad \text{on } B_{r_n}. \]

Fix \( y \). Define
\[ x_n = \sqrt{r_n^2 - y^2}. \]

By passing to a subsequence we may assume all \( r_n > |y| \) and
\[ \frac{x_n - x_{n-1}}{r_n} > \frac{1}{2} \quad \text{for all } n. \]

Then
\[ \int_0^{\infty} e^{u(x,y')} dx \geq \sum_n \int_{x_{n-1}}^{x_n} e^{u(x,y')} dx \]
\[ \geq \sum_n \frac{1}{c} \frac{x_n - x_{n-1}}{r_n} \quad (\text{by (5)}) \]
\[ = +\infty \quad (\text{by (6)}). \]

Similarly the other three integrals also equal \( +\infty \). Then Lemma 3 applies. Q.E.D.
Theorem 4. If \( u \in C^{1,\alpha}(R^2) \) \((0 < \alpha < 1)\) satisfies

\[
\liminf_{r \to \infty \land R \geq r} (\|u\|_{C^1(B_r)} - \frac{1}{2} \ln \ln R) = -\infty
\]

then \( e^udu\,dx\,dy \) is a complete Lorentzian metric on \( R^2 \).

Proof. By Corollary 1, every characteristic line is complete. Now consider an arbitrary noncharacteristic geodesic \( \gamma(t) = (x(t), y(t)) \) \((\dot{x}(t)\dot{y}(t) \neq 0)\). It suffices to prove that \( \gamma \) is forward complete. Define the notations

\[ p(t) = \dot{x}(t), q(t) = \dot{y}(t), \]
\[ \alpha(t) = \frac{1}{p(t)}, \beta(t) = \frac{1}{q(t)}, \]
\[ X(t) = (x(t), y(t), \alpha(t), \beta(t)), \]
\[ \|(x, y, z, t)\| = \sqrt{x^2 + y^2 + z^2 + t^2}, \]
\[ \|X\|_{C[0,T]} = \sup_{t \in [0,T]} \|X(t)\|. \]

The condition implies that there exist sequences

\[
(8) \quad r_n \to +\infty \quad \text{and} \quad c_n \to +\infty
\]

such that for all \( n \),

\[
\|u\|_{C^1(B_{r_n})} \leq \frac{1}{2} \ln \ln r_n - c_n.
\]

We may assume \( \{\frac{1}{2} \ln \ln r_n - c_n\} \) is an increasing sequence. Pick an increasing function \( c(r) \) such that, for all \( r \geq 0 \),

\[
(9) \quad \|u\|_{C^1(B_r)} \leq c(r),
\]

and for all \( n \),

\[
(10) \quad c(r_n) = \frac{1}{2} \ln \ln r_n - c_n.
\]

For an arbitrary \( L > 0 \), we are going to estimate

\[ T(L) = \sup\{t | X([0, t]) \text{ exists and } \|X\|_{C[0,t]} \leq L\}. \]

Note that along a geodesic \( \gamma(t) \), \( e^{u(\gamma(t))} p(t)q(t) = \text{constant} \). Hence

\[
(11) \quad p(t)q(t) = e^{u(\gamma(0)) - u(\gamma(t))} p(0)q(0).
\]

Denote

\[
(12) \quad a(L) = e^{2c(L)}|p(0)q(0)|.
\]

Then (9) and (11) show that for \( t \in [0, T(L)] \),

\[
|p(t)q(t)| \leq a(L),
\]

\[
p^2(t) + q^2(t) = p^2 q^2 (\alpha^2 + \beta^2) \leq a(L)^2 \|X(t)\|^2,
\]

\[
|u_x(\gamma(t))| + |u_y(\gamma(t))| \leq 2c(L).
\]
Since $\gamma$ is the geodesic of $e^{u}dx\,dy$, therefore

$$X(t) = X_0 + \int_0^t (p(t), q(t), u_x(\gamma(t)), u_y(\gamma(t))) \, dt,$$

where $X_0 = (x(0), y(0), \alpha(0), \beta(0))$ is the initial data. From (13)-(15), we obtain that, for $t \in [0, T(L)]$,

$$\|X(t)\| \leq \|X_0\| + \int_0^t (2c(L) + a(L)\|X(t)\|) \, dt.$$  

Let

$$f(t) = \int_0^t \|X(t)\| \, dt;$$

then (16) is

$$f'(t) \leq \|X_0\| + 2ct + af(t),$$

$$\frac{d}{dt}(e^{-at}f(t)) = e^{-at}(f' - af) \leq e^{-at}(\|X_0\| + 2ct),$$

where $a = a(L)$ and $c = c(L)$. Integrating both sides, we get

$$e^{-at}f(t) \leq \frac{2c}{a^2} + \frac{\|X_0\|}{a} - e^{-at}\left(\frac{2c}{a^2}t + \frac{2c}{a^2} + \frac{\|X_0\|}{a}\right),$$

$$f(t) \leq \left(\frac{2c}{a^2} + \frac{\|X_0\|}{a}\right)(e^{at} - 1) - \frac{2c}{a}t.$$  

Then (16)-(18) imply that, for $t \in [0, T(L)]$,

$$\|X(t)\| \leq M_L(t),$$

where $M_L \in C^\infty(R)$ is defined as

$$M_L(t) = \|X_0\| + \left(\frac{2c(L)}{a(L)} + \|X_0\|\right)(e^{a(L)t} - 1).$$

Define

$$t_n = \frac{1}{a(r_n)} \ln \frac{a(r_n)r_n + 2c(r_n)}{a(r_n)\|X_0\| + 2c(r_n)},$$

that is

$$M_{r_n}(t_n) = r_n.$$  

Then (19) shows that, as long as $X(t)$ exists,

$$\|X(t)\| \leq r_n \quad \text{for} \quad t \in [0, t_n].$$

But (8),(10), (12) and (20) imply

$$t_n \to +\infty \quad (n \to \infty).$$

Then (21) and (22) imply that there exists $L \in C(R)$ such that for all $t \geq 0$, as long as $X(t)$ exists,

$$\|X(t)\| \leq L(t).$$
Now (13) and (23) imply a priori estimate
\[ x(t)^2 + y(t)^2 + p(t)^2 + q(t)^2 \leq (1 + a(L(t))^2)L(t)^2, \]
which guarantees that $\gamma(t)$ exists for all $t \geq 0$. Q.E.D.

**Remark 4.** If $u \in C^1(R^2)$ satisfies (7) and has the following property:

For any compact subset $K \subset R^2$, there exists $\varepsilon > 0$ such that for any initial data $p \in K, v \in S^1 = \{x^2 + y^2 = 1\} \subset R^2$, the geodesic $\gamma(t)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ exists unique on $t \in [0, \varepsilon]$;

then the Lorentzian metric $e^u dx \, dy$ on $R^2$ is complete. By Peano's existence theorem in the theory of ordinary differential equations, the local existence of the geodesic is always true. But the uniqueness might be false. If $u \in C^{1,\alpha}(R^2)$ $(0 < \alpha < 1)$, then the uniqueness is also true.

**Theorem 5.** Let $f \in C^k(R^2)$ $(1 \leq k < \infty)$, 
\[ F(r) = \sup_{(x,y) \in B_r} \int_0^r (|f(x,t)| + |f(t,y)|) dt. \]

If there exist two nonnegative functions $g, h \in C(R)$ and a constant $c > 0$ such that
\[ \|p\|_{L^1(R^2)} < \infty \quad \text{where } p(x,y) = f(x,y)e^{-g(x)-h(y)}, \]
and
\[ \liminf_{r \to \infty R \geq r} \left( cF(R) + \|g\|_{C^1[-R,R]} + \|h\|_{C^1[-R,R]} - \frac{1}{2} \ln \ln R \right) = -\infty, \]
then $f$ is the curvature function for some complete $C^{k+1}$-Lorentzian metric $e^u dx \, dy$ on $R^2$.

**Proof.** Pick $\varepsilon > 0$, such that
\[ \|\varepsilon p\|_{L^1(R^2)} < \frac{1}{\pi}, \]
and
\[ \liminf_{r \to \infty R \geq r} \left( \varepsilon F(R) + \|g\|_{C^1[-R,R]} + \|h\|_{C^1[-R,R]} - \frac{1}{2} \ln \ln R \right) = -\infty. \]

By Lemma 1, there exists $w \in C^{k+1}(R^2)$ satisfying
\[ w(x,y) = -\int_0^x \int_0^y \varepsilon p(s,t)e^{w(s,t)} ds \, dt. \]
Clearly $\|w\|_{C(R^2)} \leq 1$. Then (25) implies, for all $r > 0$,
\[ \|w\|_{C^1(B_r)} \leq 1 + \varepsilon F(r). \]
Let
\[ u(x,y) = w(x,y) - g(x) - h(y). \]
Then
\[(27) \quad \|u\|_{C^1(B_r)} \leq 1 + \|g\|_{C^1([-r,r],\mathbb{R})} + \|h\|_{C^1([-r,r],\mathbb{R})} + \varepsilon F(r) . \]
Then (24), (27) and theorem 4 imply that $e^u dx dy$ is a complete Lorentzian metric. (25) and (26) imply that $K(e^u dx dy) = 2ef$. Then the complete metric $2e e^u dx dy$ has the curvature $f$. Q.E.D.

**Remark 5.** Theorem 5 also holds for $f \in C^\alpha(\mathbb{R}^2)$, $(0 < \alpha < 1)$.

Taking $g = h = 0$, we obtain the following:

**Corollary 2.** Let $f \in C^k(\mathbb{R}^2)$, $(1 \leq k \leq \infty)$. If $\|f\|_{L^1(\mathbb{R}^2)} < \infty$ and if there exists a constant $c$ such that
\[
\liminf_{r \to \infty} \left( \sup_{(x,y) \in B_R} \int_{-R}^R (|f(x,t)| + |f(t,y)|) \, dt - c \ln \ln R \right) = -\infty ,
\]
then $f$ is the curvature function for some complete $C^{k+1}$-Lorentzian metric $e^u dx dy$ on $\mathbb{R}^2$.

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**References**