COMPACT WEIGHTED COMPOSITION OPERATORS ON BANACH LATTICES

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ABSTRACT. A characterization of compact (and M-weakly compact) weighted composition operators on real and complex Banach lattices which can be appropriately realized as function spaces is provided.

The compact weighted composition operators on $C^b(X)$, all bounded continuous complex valued functions on $X$ with the supremum norm, have been characterized in the case of compact $X$ by Kamowitz [4], and for completely regular $X$ by Singh and Summers [6]. We recall that an operator $T$ on $C^b(X)$ is called a weighted composition operator if there exists an $r$ on $C^b(X)$ and a continuous map $\phi$ from $X$ to $X$ so that for each $f$ in $C^b(X)$, we have $Tf(x) = r(x)f(\phi(x))$.

In this note we will characterize a class of compact weighted composition operators (and compact composition operators) on more general function spaces, which we will call Banach $F$-lattices. These include the $L^p$-spaces and the Banach lattices, both real and complex, with quasi-interior points (more generally, topological order partitions). We also demonstrate that these operators are equivalent to the $M$-weakly compact weighted composition operators.

We will call a complex Banach space $E$ a Banach $F$-lattice if there exists a real Banach lattice $G$, which can be identified with equivalence classes of real valued functions on a completely regular space $X$, and $E = G + iG$ can be identified with equivalence classes of complex valued functions on $X$ satisfying the following conditions:

(i) If $f$ is equivalent to $g$, then $\{x \in X : f(x) = g(x)\}$ is dense.

(ii) Each continuous complex valued bounded function on $X$ (or function with compact support in the case of a locally compact $X$) represents an element in $E$.

(iii) The vector space operations on $E$ and the lattice operations on $G$ correspond to the pointwise defined operations.
(iv) For each $f$ in $E$, the function $|f|(x) = |f(x)|$ is a well-defined element in $E$ and $\|f\| = \|\ |f|\|$.

(v) Given $h \geq 0$, a bounded continuous real valued function on $X$ and $f$ in $E$, the pointwise product $hf$ is a well-defined element of $E$.

The $L^p$ spaces are examples of Banach $F$-lattices. We will verify that Banach lattices (complex as well as real) with locally compact representation spaces are also Banach $F$-lattices.

We recall (see [2] or [5]) that a real Banach lattice $G$ is said to have a locally compact representation space $X$ if the space $C_k(X, \mathbb{R})$, all real-valued continuous functions on $X$ with compact support, can be identified with a dense (order) ideal in $G$. In fact, $G$ has a locally compact representation space if and only if $G$ contains a topological order partition (see [2]). We will say that a complex Banach lattice (in the sense of Schaefer [5]) has a locally compact representation space $X$ if $E = G + iG$ and $G$ has a locally compact representation space $X$.

Given that $E$ has a locally compact representation space $X$, the real Banach lattice $G$ can be identified with continuous extended real-valued functions on $X$ (i.e., range the two point compactification of $\mathbb{R}$), each finite on a dense subset. Thus $E$ can be identified with continuous functions from $X$ to the one point compactification of $\mathbb{C}$, each finite on a dense subset of $X$. Now for $f$ in $E$, a complex valued function $g$ will be said to be equivalent to $f$ if \( \{ x \in X : f(x) = g(x) \} \) is a dense open subset of $X$. We will now establish that $E$ is a Banach $F$-lattice via the identification of each $f$ in $E$ with this equivalence class of complex functions on $X$.

**Lemma.** Let $E$ be a complex Banach lattice with locally compact representation space $X$. Then $E$ is a Banach $F$-lattice.

**Proof.** We note that condition (iv) follows from the definition of a complex Banach lattice (see [5, p. 137]) and thus we need only verify that (v) is satisfied. Let $h \geq 0$ be a bounded continuous real-valued function on $X$ and let $f$ be in $E$. Then $f = f_1 + if_2$ as a function on $X$ where $f_1$ and $f_2$ are real-valued. We view $f_1$ as a continuous function from $X$ to the two point compactification of $\mathbb{R}$ and choose an increasing net $(f_\alpha)$ of functions in $C_k(X, \mathbb{R})$ converging up to $f_1$ in norm and pointwise for all $x$. Now $(hf_\alpha)$ is a Cauchy net since $|hf_\alpha - hf_\beta| \leq \|h\|_\infty |f_\alpha - f_\beta|$ so that $\|hf_\alpha - hf_\beta\| \leq \|h\|_\infty \|f_\alpha - f_\beta\|$. Hence $(hf_\alpha)$ converges to some element $g$ in $G$ which we identify with an extended real valued function. If there exists an $x_0$ in $X$ such that $g(x_0) > hf_1(x_0)$ then $g$ exceeds $hf_1$ on a compact neighborhood $N$ of $x_0$. Let $k \geq 0$ be in $C_k(X, \mathbb{R})$ such that $k$ vanishes off of $N$, and $(g - hf_1(x)) \geq k(x)$ for all $x$. Now $(g - hf_\alpha)$ exceeds $k$ for all $x$ and $\alpha$, so that $\|g - hf_\alpha\| \geq \|k\|$, a contradiction. Hence $g \leq hf_1$. If there exists an $x_0$ such that $g(x_0) < hf_1(x_0)$, then $g(x_0) < hf_\alpha(x_0)$ for some $\alpha$. Again $g < hf_\alpha$ on a neighborhood $N$ of $x_0$, and we choose a function $k \geq 0$ with $k < (hf_\alpha - g)$. Thus $\|k\| \leq \|hf_\alpha - g\|$. 

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for all \( \alpha > \hat{\alpha} \), and we conclude that \( h f_1 = g \). A similar argument applies to \( f_2 \).

Given \( E \) and \( F \) Banach \( F \)-lattices identified with equivalence classes of complex valued functions on \( X \) and \( Y \) respectively, we will call a bounded operator \( \Phi \) from \( E \) to \( F \) a composition operator if there exists a continuous function \( \phi \) from \( Y \) to \( X \) so that

\[
\Phi(f)(y) = f(\phi(y))
\]

for each \( f \) in \( E \) and \( y \) in \( Y \). A bounded operator \( T \) will be called a weighted composition operator if there exists a continuous complex valued function \( r \) on \( Y \) and a composition operator \( \Phi \) that

\[
T(f)(y) = r(y)\Phi(f)(y) = r(y)f(\phi(y))
\]

We will often write \( T = r(\phi \circ \Phi) \).

A weighted composition operator \( T \) will be said to satisfy condition \((*)\) if there exists a \( \delta \geq 0 \) so that for each \( x \) in \( \phi(Y) \) and neighborhood \( N \) of \( x \) there is an element \( f \) in \( E \) with \( ||f|| = 1 \) equivalent to a function vanishing outside of \( N \), and \( ||\Phi(f)|| \geq \delta \).

Although many operators will satisfy condition \((*)\), for example, translations or shifts, we cite an elementary example where this condition fails. On \( L^1[0,1] \), let \( \Phi(f) = f(\sqrt{x}) \) (\( \phi(x) = \sqrt{x} \)). At \( x = 0 \) condition \((*)\) is not satisfied.

**Theorem 1.** Let \( E \) and \( F \) be Banach \( F \)-lattices identified with equivalence classes of functions on \( X \) and \( Y \) respectively, and \( T \) a weighted composition operator from \( E \) to \( F \). Let \( r \) be in \( C(Y) \), and \( \phi \) a continuous map from \( Y \) to \( X \), so that \( Tf = r(f \circ \phi) \) for each \( f \) in \( E \). If \( T \) satisfies condition \((*)\) and for each \( \varepsilon > 0 \), there is a positive \( \delta \) less than \( \varepsilon \), so that \( \phi\{y: |r(y)| \leq \delta\} \) is disjoint from \( \phi\{y: |r(y)| \geq \varepsilon\} \), then the following statements are equivalent:

(i) \( T \) is a compact operator.

(ii) For each \( \varepsilon > 0 \), the set \( \phi\{y: |r(y)| > \varepsilon\} \) is finite.

**Proof.** We first assume that condition (ii) is not satisfied. Thus there exists an \( \varepsilon > 0 \) and a sequence \( \{a_i\} \) of distinct points in the image of \( \{y: |r(y)| \geq \varepsilon\} \) under the map \( \phi \). We construct a collection of disjoint closed neighborhoods for an infinite collection of points in \( \{a_i\} \) as follows. Choosing a subsequence if necessary, we assume that no subsequence of \( \{a_i\} \) converges to a point in \( \{a_i\} \). Now let \( N_1 \) be a neighborhood of \( a_1 \) such that \( \{a_1\} - N_1 \) is an infinite set and \( N_1 \) is disjoint from the set \( F = \phi\{y: |r(y)| < \varepsilon\} \). Let \( a_2 \) be a point in \( \{a_1\} - N_1 \) and choose a closed neighborhood \( N_2 \) of \( a_2 \) disjoint from \( N_1 \) and \( F \) such that \( \{a_1\} - (N_1 \cup N_2) \) is an infinite set. Continuing inductively, let \( N_i \) be a neighborhood of a point \( a_i \), where \( a_i \) is in the infinite set \( \{a_i\} - \bigcup_{k<j} N_k \), disjoint from \( \bigcup_{k<j} N_k \) and \( F \) such that \( \{a_i\} - \bigcup_{k<j} N_k \) is infinite. Now condition \((*)\) implies that for each \( i \) there exists an \( f_i \) in \( E \) with \( f_i \)
equal to zero on the complement of $N_i$, $\|f_i\| = 1$, and $\|\Phi(f_i)\| \geq \delta$. Now for $j \neq k$,

$$\|Tf_j - Tf_k\| = \|r(f_j \circ \phi) - r(f_k \circ \phi)\| \geq \|r(f_j \circ \phi)\| \geq \delta \|\Phi(f_j)\| \geq \delta \delta$$

since $f_j$ and $f_k$ have disjoint support and $|r|$ exceeds $\delta$ on the points where $(f_j \circ \phi)$ is nonzero. Thus the sequence $(Tf_j)$ has no convergent subsequence.

Assume that (ii) is satisfied. Given $\epsilon > 0$, let $H$ be a positive continuous real-valued function on $Y$ which is one on $\{y: |r(y)| \geq \epsilon\}$ and zero on $\{y: |r(y)| < \epsilon/2\}$, and bounded by one, (e.g., $2/\epsilon(|r| - \frac{1}{2}\epsilon)^+$. Now for $f$ in the unit ball of $E$

$$\|HTf - Tf\| \leq \|f \circ \phi\| \leq \epsilon \|\Phi\|,$$

where $\|\Phi\|$ is the operator norm of $\Phi$. We will complete the proof by showing that $HT$ is a finite rank operator (i.e., $T$ is the limit of finite rank operators). Let $\phi(y: |r(y)| \geq \epsilon/2) = \{a_i: i = 1, 2, \ldots, n\}$ and for each $i$ choose a continuous bounded real-valued function $L_i$ so that $L_i(a_j) = 1$ and $L_i(a_j) = 0$ for $i \neq j$. Now for each $f$ and $i$, there exists a constant $\alpha_i(f)$ so that

$$H(y)r(y)(L_i \circ \phi)(y)(f \circ \phi)(y) = \alpha_i(f)H(y)r(y)(L_i \circ \phi)(y)$$

for all $y$ in $Y$. Hence $HTf = \sum_{i=1}^{n} \alpha_i(f)Hr(L_i \circ \phi)$.

Corollary. Let $E$ and $F$ be Banach $\ell_1$-lattices of functions on $X$ and $Y$ respectively. Let $\Phi$ be a composition operator such that $\Phi(f) = f \circ \phi$ for $\phi$ a continuous map from $Y$ to $X$ and $f$ in $E$. If $\Phi$ satisfies condition $(\ast)$, then $\Phi$ is compact if and only if the image of $Y$ under $\phi$ is finite.

In the special case that $E$ and $F$ are spaces of all continuous bounded functions on a completely regular space with the sup-norm topology, the $(\ast)$ assumption and the separation assumption are not necessary in the previous results. Thus we provide an alternative proof for the result established by Singh and Summers in [6], and, in a slightly different form, by Kamowitz for compact $X$ in [4].

Theorem 2. Let $C^b(X)$ and $C^b(Y)$ be the spaces of all continuous complex valued bounded functions on $X$ and $Y$ respectively with the sup-norm topology. Let $\phi$ be a continuous map from $Y$ to $X$ and $r$ a continuous bounded complex valued function on $Y$.

(a) The composition operator $\Phi$ defined by $\Phi(f) = f \circ \phi$ for each $f$ in $C^b(X)$ is compact if and only if the image of $Y$ under $\phi$ is finite.

(b) The weighted composition operator $T$ defined by $Tf = r(f \circ \phi)$ for each $f$ in $C^b(X)$ is compact if and only if for each $\epsilon > 0$, the image under $\phi$ of $\{y: |r(y)| \geq \epsilon\}$ is finite.

Proof. We need only prove part (b) and proceed as in the proof of Theorem 1. Assume first that there is an infinite sequence of distinct points $(a_i)$ in the
image of \( \{ y : r(y) \geq \varepsilon \} \) under the map \( \phi \). We choose neighborhoods \( N_i \) as in the proof of Theorem 1 without regard to the set \( F \) (i.e., without \( N_i \) disjoint from \( F \)). Then the function \( f_j \) is chosen to be one at \( a_{ij} \) and zero on the complement of \( N_i \) with \( 0 \leq f_j \leq 1 \). For \( y \) in \( Y \) with \( r(y) \geq \varepsilon \) and \( \phi(y) = a_{ij} \), we have

\[
\| T f_j - T f_k \| \geq \| r(f_j \circ \phi) - r(f_k \circ \phi) \| \geq |r(y) f_j (a_{ij})| = |r(y)| \geq \varepsilon.
\]

The proof of the converse is the same as the proof in Theorem 1.

We recall that (e.g., see [1, p. 313]) that an operator from \( E \) to \( F \) is called \( M \)-weakly compact if for every disjoint sequence \( (f_n) \) in \( E \) bounded in norm, the sequence \( (\| T(f_n) \|) \) converges to zero (\( f \) is disjoint from \( g \) if \( |f|A|g| = 0 \)).

We can now prove the following:

**Theorem 3.** A weighted composition operator \( T \) satisfying the hypothesis of Theorem 1 is compact if and only if \( T \) is \( M \)-weakly compact.

**Proof.** We will verify that \( T \) is \( M \)-weakly compact if and only if condition (ii) of Theorem 1 is satisfied. Given that (ii) is not satisfied, the same argument as in the proof of Theorem 1 yields a bounded disjoint sequence whose image under \( T \) does not converge to zero in norm. Given that (ii) is satisfied, let \( H \) be the function in the proof of Theorem 1. We note that for \( (f_j) \) a disjoint sequence bounded in \( E \), the sequence \( (HT f_j) \) is also disjoint. Since \( \phi \{ y : r(y) \geq \varepsilon/2 \} \) is finite, at most finitely many of the functions \( HT f_j \) can be nonzero. Thus \( (HT f_j) \) is convergent to zero in norm. Now \( HT \) is \( M \)-weakly compact and since the \( M \)-weakly compact operators are closed (e.g., [1, p. 317]), we conclude that \( T \) is \( M \)-weakly compact.

**References**


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