YEKT ANOTHER PROOF OF
THE LYAPUNOV CONVEXITY THEOREM

ZVI ARTSTEIN

(Communicated by R. Daniel Mauldin)

Abstract. A new proof is given, of the convexity and compactness of the range of an atomless $R^n$-valued measure.

Several proofs are available for the theorem of A. A. Lyapunov on the range of a vector measure. (The bibliography given here is not exhaustive.) These prove reflect both the applicability and the value of the theorem. This paper presents yet another proof, one based on a new, useful argument.

The measure theory we use is standard. Let $(\Omega, \Sigma)$ be a measurable space, and let $\mu = (\mu_1, \ldots, \mu_n)$ be an atomless $R^n$-valued $\sigma$-additive finite measure on it. The range of the restriction of $\mu$ to a set $T$ in $\Sigma$ is

$$R(T) = \{\mu(A) : A \subset T, A \in \Sigma\}.$$ 

We denote by $|\mu|$ the scalar measure of total variation of $\mu$. From here on we identify sets which differ by only a set of $|\mu|$-measure zero. Thus $T_1 \subset T_2$ means that $|\mu|(T_1 \setminus T_2) = 0$. We denote by $chK$ the closed convex hull of the set $K \subset R^n$. With this notation the Lyapunov theorem reads $chR(\Omega) = R(\Omega)$.

We arrive at it as the conclusion of the following result.

Theorem. Let $x$ be in $chR(\Omega)$. Consider the subclass $\Sigma^1$ of $\Sigma$, consisting of those $T \in \Sigma$ such that $x \in chR(T)$. Then $\Sigma^1$ contains a minimal set, say $S$, with respect to inclusion (minimal up to $|\mu|$-null set). For the minimal set $S$ we have $\mu(S) = x$. In particular $x \in R(\Omega)$, and the latter is therefore closed and convex.

We use the following result.

Lemma. Let $T = \bigcap_{i=1}^{\infty} T_i$, where $T_1 \supset T_2 \supset \cdots$ is a decreasing sequence in $\Sigma$. Then $chR(T) = \bigcap_{i=1}^{\infty} chR(T_i)$.

Proof. The inclusion of $ch(R(T))$ in the intersection is trivial. To verify the other direction, and since all the sets are compact, it suffices to prove that if $y_i \in chR(T_i)$ then the distance between $y_i$ and $ch(R(T))$ tends to zero as $i \to \infty$.

Received by the editors May 24, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 28B05.
$i \to \infty$. Since the closure and taking convex hull operations do not increase the distance from the convex set $\text{ch}R(T)$, it is enough to verify the claim for $y_i \in R(T_i)$, namely when $y_i = \mu(A_i)$ for $A_i \subset T_i$. In particular $y_i = \mu(A_i \cap T) + \mu(A_i \setminus T)$. The first term belongs to $R(T)$; the second term is bounded in norm by $|\mu|(T \setminus T)$. The latter sequence converges to zero (an elementary fact of scalar measures, implied by the $\sigma$-additivity); hence the vectors $y_i - \mu(A_i \cap T)$ tend to zero and this verifies the claim.

**Proof of the existence of a minimal element in $\Sigma^1$.** Let $T_\gamma, \gamma \in \Gamma$, be a decreasing family, not necessarily countable, of sets in $\Sigma^1$. We claim that a cofinal subsequence $T_\gamma_i, i = 1, 2, \ldots,$ exists; namely, each $T_\gamma_i$ contains an element of the sequence. To show this, consider the numbers $|\mu|(T_\gamma_i), \gamma \in \Gamma$, and choose a sequence $|\mu|(T_\gamma_i)$ among these numbers such that each $|\mu|(T_\gamma_i)$ is greater than or equal to one of the elements in the sequence; $T_\gamma_i$ is then cofinal. Clearly, $T = \bigcap_{i=1}^{\infty} T_\gamma_i$ is included (up to $|\mu|$-null sets) in each $T_\gamma_i$. By the lemma, if each $T_\gamma_i$ belongs to $\Sigma^1$, then $T \in \Sigma^1$; i.e., $T_\gamma, \gamma \in \Gamma$, has a lower bound in $\Sigma^1$. By the Zorn lemma a minimal element exists.

**Some notations.** Let $p \cdot x$ denote the scalar product of $p$ and $x$ in $\mathbb{R}^n$. If $K \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$, then $K_p$ is the $p$-boundary of $K$; namely $K_p = \{y \in K: p \cdot y \geq p \cdot z \text{ for all } z \in K\}$. For $K \subset \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we write $y + K$ for $\{y + z: z \in K\}$. We fix $p \in \mathbb{R}^n$. Note that the set function $p \cdot \mu$, defined by $(p \cdot \mu)(A) = p \cdot \mu(A)$, is a $\sigma$-additive signed measure. For $T \in \Sigma$ we denote by $T_+, T_-$, and $T_0$ the decomposition of $T$ into sets, such that $p \cdot \mu$ is nonnegative on subsets of $T_+$ and nonpositive on subsets of $T_-$, and such that $|p \cdot \mu|$ vanishes on $T_0$ and $T_0$ is maximal in the sense that $|p \cdot \mu|(A) = 0$ then $|\mu|(A \setminus T_0) = 0$ (namely $|\mu|$ is absolutely continuous with respect to $p \cdot \mu$ on $T_+ \cup T_-$). It is easy to construct this decomposition (e.g. if $f(w)$ is the Radon-Nikodym derivative of $\mu$ with respect to $|\mu|$, then $T_+ = \{w \in T: p \cdot f(w) > 0\}$, etc.).

**Proposition.** Let $T \in \Sigma$. Then $(\text{ch}R(T))_p = \mu(T_+) + \text{ch}R(T_0)$.

**Proof.** The inclusion $\mu(T_+) + \text{ch}R(T_0)$ in the $p$-boundary of $\text{ch}R(T)$ is trivial. To verify the other direction, let $y \in (\text{ch}R(T))_p$; we have to show that $y \in \mu(T_+) + \text{ch}R(T_0)$. Since for bounded sets the closure operation and taking convex-hull operation commute, it is enough to verify the inclusion for $y$ in the closure of $R(T)$; namely when $y = \lim \mu(T_j)$ and $T_j \subset T$. We claim that for $|\mu|(T_+ \setminus T_j)$ and $|\mu|(T_- \cap T_j)$, both converge to zero as $j \to \infty$. This follows immediately from the convergence of $p \cdot \mu(T_j)$ to $p \cdot y = \max\{p \cdot z: z \in \text{ch}R(T)\}$, and the splitting of $T$ into the positive, negative, and neutral parts with respect to $p \cdot \mu$. Once the convergence to zero of $|\mu|(T_- \cap T_j)$ and $|\mu|(T_+ \setminus T_j)$ is established, we notice that $y$ is also the limit of $\mu(T_+) + \mu(T_0 \cap T_j)$. The latter sequence is in $\mu(T_+) + \text{ch}R(T_0)$, and this is what we have to show.
Proof of the equality $x = \mu(S)$.

Case 1. $x$ is in the relative interior of $\text{ch}R(S)$. Since the latter contains the zero vector, it follows that $x$ would also be in $\text{ch}R(S^1)$ if $|\mu|(S \setminus S^1)$ is small enough. Such an $S^1$ with $|\mu|(S \setminus S^1) > 0$ is easily constructed by the lack of atoms of $|\mu|$. This contradicts the minimality of $S$; thus $x$ cannot be in the relative interior of $\text{ch}R(S)$.

Case 2. $x$ is in the relative boundary of $\text{ch}R(S)$. Then a $p \in \mathbb{R}^n$ exists with $x \in (\text{ch}R(S))^p$ and $p \cdot x > p \cdot y$ for some $y \in R(S)$. By the proposition, $x - \mu(S_+^+) \in \text{ch}R(S_0)$, where $S_+^+$ and $S_0$ are defined with respect to $p$. Clearly $S_0$ is a minimal set with this property; otherwise minimality of $S$ is contradicted. The linear dimensionality of $\text{ch}R(S_0)$ is smaller than that of $\text{ch}R(S)$; thus an induction argument (or repeating the argument $n - 1$ times) completes the proof.

Bibliography


Department of Theoretical Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel