Let $\mathbb{Q}_p$ be the $p$-adic completion of $\mathbb{Q}$ for a prime $p$. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ be its $p$-adic completion which is an algebraically closed field of cardinality $2^{\aleph_0}$. Let $\mathbb{Q}_p^{\text{unram}}$ be the maximum unramified extension field of $\mathbb{Q}_p$. Then $\mathbb{Q}_p^{\text{unram}} = \mathbb{Q}_p(W)$, where $W$ is the set of all roots of unity whose orders are prime to $p$. Let $\mathbb{C}_p^{\text{unram}}$ be the $p$-adic closure of $\mathbb{Q}_p^{\text{unram}}$ in $\mathbb{C}_p$. Koblitz [1] asked whether $\mathbb{C}_p^{\text{unram}}$ has uncountably infinite transcendence degree over $\mathbb{Q}_p$ and $\mathbb{C}_p$ has uncountably infinite transcendence degree over $\mathbb{C}_p^{\text{unram}}$. Lampert [2] answered that the transcendence degree of $\mathbb{C}_p^{\text{unram}}$ over $\mathbb{Q}_p$ is $2^{\aleph_0}$ and the transcendence degree of $\mathbb{C}_p$ over $\mathbb{C}_p^{\text{unram}}$ is $2^{\aleph_0}$ by constructing sets of algebraically independent numbers of cardinality $2^{\aleph_0}$. Here we will give more explicit examples of such sets which cannot be obtained by the method in [2].

**Theorem.** Let $K$ be an intermediate field between $\mathbb{Q}_p$ and $\mathbb{C}_p$. Let $\alpha_1, \ldots, \alpha_m$ be in $\mathbb{C}_p$ and $\alpha_1, \ldots, \alpha_{m-1}$ be algebraically independent over $K$. Suppose that for $i = 1, \ldots, m-1$ there exist sequences $\{\beta_{ik}\}_{k \geq 1}$ in $\mathbb{C}_p$ converging to $\alpha_i$ and a sequence $\{S_k\}_{k \geq 1}$ of finite subsets of $\text{Aut}(\mathbb{C}_p/K(\{\beta_{ik}\}_{1 \leq i \leq m-1}))$ which satisfies

\[
\lim_{k \to \infty} |S_k| = \infty \quad \text{and} \quad \alpha_m^\sigma \neq \alpha_m^\tau \quad \text{for any} \quad \sigma, \tau \in S_k \text{ with } \sigma \neq \tau,
\]

\[
\max_{1 \leq i \leq m-1} |\alpha_i - \beta_{ik}|_p = o \left( \min_{\sigma, \tau \in S_k} |\alpha_m^\sigma - \alpha_m^\tau|_p \right) \quad \text{as } k \to \infty,
\]

where we define the left-hand side of (2) to be $0$ if $m = 1$. Then $\alpha_1, \ldots, \alpha_m$ are algebraically independent over $K$.

To prove the theorem we need the following lemma which is proved in Koblitz [1].
Lemma (Koblitz [1], p. 70). Let \( f(X) \in \mathbb{C}_p[X] \) have degree \( n \),
\[
f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0.
\]
Suppose that \( f(X) \) has no multiple root. Then there exists a positive constant \( c \) such that if \( g(X) = \sum_{i=0}^{n} b_iX^i \in \mathbb{C}_p[X] \) has degree \( n \), and if \( \max_{0 \leq i \leq n} |a_i - b_i|_p \) is sufficiently small, then for every root \( \beta \) of \( g(X) \) there is precisely one root \( \alpha \) of \( f(X) \) such that
\[
|\alpha - \beta|_p \leq \max_{1 \leq i \leq n} |a_i - b_i|_p.
\]

Proof of theorem. Suppose that \( \alpha_1, \ldots, \alpha_m \) are algebraically dependent over \( K \). Then there exists a polynomial \( f(X) \) of degree \( n \) with coefficients in \( K[\alpha_1, \ldots, \alpha_{m-1}] \),
\[
f(X) = Q_n(\alpha_1, \ldots, \alpha_{m-1})X^n + \cdots + Q_0(\alpha_1, \ldots, \alpha_{m-1})
\]
such that \( f(\alpha_m) = 0 \) and \( f(X) \) has no multiple root. If \( \sigma \in S_k \), then
\[
|Q_i(\alpha_1^\sigma, \ldots, \alpha_{m-1}^\sigma) - Q_i(\alpha_1, \ldots, \alpha_{m-1})|_p 
\leq \max\{|Q_i(\alpha_1^\sigma, \ldots, \alpha_{m-1}^\sigma) - Q_i(\beta_1^k, \ldots, \beta_{m-1,k})|_p, 
|Q_i(\beta_1^k, \ldots, \beta_{m-1,k}) - Q_i(\alpha_1, \ldots, \alpha_{m})|_p\}
\leq c_1 \max_{1 \leq i \leq m-1} |\alpha_i - \beta_{ik}|_p,
\]
where \( c_1 \) is a positive constant. If \( k \) is sufficiently large, then \( |S_k| > n \) and by the lemma, there exists a root \( \alpha \) of \( f(X) \) and two distinct elements \( \sigma, \tau \) of \( S_k \) such that
\[
|\alpha - \alpha_m^\sigma|_p, |\alpha - \alpha_m^\tau|_p \leq c_2 \max_{1 \leq i \leq m-1} |\alpha_i - \beta_{ik}|_p,
\]
where \( c_2 \) is a positive constant, and so
\[
\min_{\sigma, \tau \in S_k} |\alpha_m^\sigma - \alpha_m^\tau|_p \leq c_2 \max_{1 \leq i \leq m-1} |\alpha_i - \beta_{ik}|_p.
\]
This contradicts condition (2) and the theorem is proved.

It is well known that every element \( \alpha \) of \( \mathbb{C}^{\text{unram}}_p \) is uniquely represented as \( \alpha = \sum_{n \geq q} \zeta np^n \) where \( \zeta \in W \) and \( q \in \mathbb{Z} \). The number \( \alpha \) is transcendental over \( \mathbb{Q}_p \) if and only if the extension degree \( [\mathbb{Q}_p(\zeta^n) : \mathbb{Q}_p] \), \( n \geq q \), is unbounded. By using the theorem, we obtain a set of cardinality \( 2^{\aleph_0} \) whose elements are in \( \mathbb{C}^{\text{unram}}_p \) and algebraically independent over \( \mathbb{Q}_p \).

Example 1. Let \( \zeta(n) \) be a primitive \( n \) th root of unity for every natural number \( n \). Let \( P \) be the set of all prime numbers. Then the numbers
\[
\sum_{n=1}^{\infty} \zeta(l^{\lambda n})p^n, \quad (l \in P - \{p\}, \lambda \in \mathbb{R}^+)
\]
are algebraically independent over \( \mathbb{Q}_p \).
Proof. Let \( l_1, \ldots, l_s \in P - \{p\} \) and \( K \) be the \( p \)-adic closure of \( \mathbb{Q}_p(\{\zeta(l_i^n)\}_{1 \leq i \leq s, n \geq 0}) \). Let \( l \in P - \{p, l_1, \ldots, l_s\} \) and \( 0 < \lambda_1 < \cdots < \lambda_m \). Put
\[
\alpha_i = \sum_{n=0}^{\infty} \zeta(l_i^{\lambda_n}) p^n, \quad 1 \leq i \leq m.
\]
It is enough to prove that \( \alpha_1, \ldots, \alpha_m \) are algebraically independent over \( K \). We prove it by induction on \( m \). Assume that \( \alpha_1, \ldots, \alpha_{m-1} \) are algebraically independent over \( K \). Put
\[
\beta_{ik} = \sum_{n=1}^{k + \lfloor \log k \rfloor} \zeta(l_i^{\lambda_n}) p^n, \quad 1 \leq i \leq m - 1, \ k \geq 1,
\]
and
\[
d_k = [K(\zeta(l_1^{\lambda_m k})) : K(\zeta(l_1^{\lambda_{m-1}(k + \lfloor \log k \rfloor)))]].
\]
Then
\[
|\alpha_i - \beta_{ik}|_p = p^{-k - \lfloor \log k \rfloor - 1}
\]
and \( \lim_{k \to \infty} d_k = \infty \). Let \( S_k \) be a set of \( d_k \) isomorphisms of \( C_p \) which is obtained by extending \( \text{Gal}(K(\zeta(l_1^{\lambda_m k}))/K(\zeta(l_1^{\lambda_{m-1}(k + \lfloor \log k \rfloor}))). \) Then
\[
\min_{\sigma, \tau \in S_k} |\alpha_m^\sigma - \alpha_m^\tau|_p \geq p^{-k}.
\]
Hence by the theorem, \( \alpha_1, \ldots, \alpha_m \) are algebraically independent over \( K \).

In a similar way, we obtain a set of cardinality \( 2^{\aleph_0} \) whose elements are in \( C_p \) and algebraically independent over \( C_p^{\text{unram}} \).

Example 2. The numbers
\[
\sum_{n=1}^{\infty} p^{n+\lambda n}, \quad (l \in P - \{p\}, \ \lambda \in \mathbb{R}^+)
\]
are algebraically independent over \( C_p^{\text{unram}} \).

References


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