

ZEROS OF DIFFERENTIALS ALONG ONE-FIBERED IDEALS

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ABSTRACT. Let (R, m) be a complete local domain containing the rationals. If $I \subseteq R$ is a one-fibered ideal then there is a constant l , depending only on R and I , such that if $f \in m$ and $f \notin I^n$, then there exists a derivation d such that $d(f) \notin I^{n+l}$.

M. Hochster and C. Huneke have recently introduced the notion of tight closure of ideals and submodules. They proved, under some mild conditions, that an isolated singularity in which every ideal generated by a system of parameters is tightly closed is a rational singularity in characteristic 0 ([HH, (4.2), (4.1)]). It is conjectured that the converse of this statement holds true as well. Using an idea employed by Fedder in [F] it would be a key point in the proof of the converse to prove the following statement:

Let (R, m) be a complete local domain containing a field, let $K \subseteq R$ be a coefficient field, and assume that K is perfect. Furthermore let $I \subseteq R$ be a one-fibered ideal (see Definition 1.1 below). Then there exists an $l = l(R, I)$, depending on R and I only, satisfying the following condition:

If $f \in R$ with $\delta(f) \in I^n$ for all derivations $\delta: R \rightarrow R$ then there exists a $g \in R$ such that $f + g \in I^{n-l}$ and $d(g) = 0$, where

$$d: R \rightarrow \tilde{\Omega}_{R/K}^1$$

is the universally finite derivation of R/K ([KD, §11]).

In this note we show that this statement holds true if $\text{char}(K) = 0$, in which case $g = 0$ if $f \in m$. In characteristic p , in the graded case, Fedder has some partial results on this question (cf. [F]).

In dimension 2, all rational singularities possess one-fibered ideals, so in the context of trying to show that system of parameters are tightly closed for isolated singularities which are rational, it is reasonable to assume the existence of such ideals. Not all local rings have such ideals, however (cf. [C]). All rings in this paper are commutative and noetherian.

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Let (R, m) be a local domain and let $I \subseteq R$ be an m -primary ideal. Denote by $\mathcal{R}(I) = R[[t]]$ the corresponding Rees-ring, and by $\overline{\mathcal{R}(I)}$ its normalization.

Definition 1.1 ([S]). The ideal I is called *one-fibered* if $\text{rad}(I\overline{\mathcal{R}(I)})$ is a prime ideal, i.e., if I has only one associated Rees-valuation. (See [Mc, Ch.XI] for the definition of Rees valuations.)

Remark 1.2. One can characterize one-fibered ideals in several ways. Samuel defined the following function: $v_I(x) =$ largest integer n such that $x \in I^n$. This function is the “order” function of R along I . Define $\bar{v}_I(x) = \lim_{n \rightarrow \infty} v_I(x^n)/n$. Samuel [Sa] showed this limit exists while Rees [R2] proved that $\bar{v}_I(x)$ is a rational number and $\bar{v}_I(x) \geq n$ iff $x \in \overline{I^n}$, where for any ideal $J \subseteq R$, the integral closure of J , \bar{J} , is the set of elements $x \in R$ which satisfy an equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0,$$

where $a_i \in J^i$, $1 \leq i \leq n$ (cf. [P]).

It is known (cf. [S]) that I is one-fibered iff \bar{v}_I is a discrete valuation.

Suppose that R is Noetherian, local, analytically unramified and that $\overline{I^n}$ are valuation ideals for some rank 1 discrete valuation v for all n . Then I is necessarily one-fibered. In fact, we will show that $\bar{v}_I(x) = v(x)/v(I)$, which implies that \bar{v}_I is a valuation. Let $v(x) = k, n = v(I)$. Then $v(x^{nl}) = v(I)lk$, so that $x^{nl} \in I^{lk} \setminus I^{lk+1}$. As R is analytically unramified, there is a constant c such that $\overline{I^n} \subseteq I^{n-c}$. Hence, $x^{nl} \notin I^{lk+1}$, but $x^{nl} \in I^{lk-c}$, for all l . Thus $(lk - c)/ln \leq v_I(x^{nl})/nl < (lk + 1)/ln$. Taking limits as $l \rightarrow \infty$ gives that $v(x)/v(I) = k/n = \bar{v}_I(x)$, as required.

Lemma 1.3. *Let (R, m) be an analytically unramified one-dimensional local domain with integral closure S . Then the following conditions are equivalent:*

- (i) S is local.
- (ii) There exists a 1-fibered ideal $I \subseteq R$.
- (iii) Every nonzero ideal $I \subseteq R$ is 1-fibered.

Proof. Clearly (iii) implies (ii). We will show first that (ii) implies (i). By [S, Prop., p. 4], R is analytically irreducible. If S was not local, it would have at least two maximal ideals, which forces \hat{R} to have at least two minimal primes. This contradiction shows that S is local, which proves (i).

Next we show that (i) implies (iii). Since S is local, S is a rank 1 discrete valuation ring (DVR). Furthermore, if $I \subseteq R$, then $\overline{I^n} = I^n S \cap R$ since S is the integral closure of R . Hence $\overline{I^n}$ are valuation ideals for the DVR S , and by Remark 1.2 above this implies that I is one-fibered. \square

A key point in the proof of the theorem mentioned above is the following result:

Theorem 1.4. *Let (R, m) be a d -dimensional analytically unramified domain, let $I \subseteq R$ be a one-fibered m -primary ideal, and suppose that R has an infinite residue field $k = R/m$. Then there exists an $l = l(R, I)$ satisfying the following:*

If $f \in m$ and $f \notin I^n$ then there exist $g_2, \dots, g_d \in I$ such that $(f, g_2, \dots, g_d) \supseteq I^{n+l}$.

Proof. Case 1. $d = 1$. Note that the existence of a one-fibered ideal implies that R is analytically irreducible ([S, Prop., p. 4]), i.e., that \widehat{R} , the m -adic completion of R , is a domain. By the faithful flatness of \widehat{R}/R it obviously suffices to show that there exists an l such that whenever $f \in m$, $f \notin I^n$ then $f\widehat{R} \supseteq I^{n+l}\widehat{R}$. Let $S = \overline{\widehat{R}}$ be the integral closure of \widehat{R} in the quotient field K of \widehat{R} . Then S is a finite \widehat{R} -module since \widehat{R} is Nagata, and S is local. By the Artin-Rees Lemma there exists an $l_1 \in \mathbb{N}$ such that $I^{n+l_1}S \cap \widehat{R} = I^n(\widehat{R} \cap I^{l_1}S) \subseteq I^n\widehat{R}$ for all n . So from $f \notin I^n$ it follows $f \notin I^{n+l_1}S$. Since S is a DVR we get $IS = xS$ for some $x \in S$, and furthermore $fS \supseteq I^{n+l_1}S$.

Denote by $\mathcal{L}_{S/\widehat{R}}$ the conductor from S to \widehat{R} . Since S is a finite \widehat{R} -module $\mathcal{L}_{S/\widehat{R}}$ is an m -primary ideal, and therefore there exists an $l_2 \in \mathbb{N}$ such that $I^{l_2}\widehat{R} \subseteq \mathcal{L}_{S/\widehat{R}}$. Then for $l = l_1 + l_2$ it holds:

$$f\widehat{R} \supseteq f\mathcal{L}_{S/\widehat{R}}S \supseteq fI^{l_2}S \supseteq I^{l_2}I^{n+l_1}S = I^{n+l}S \supseteq I^{n+l}\widehat{R}$$

as desired.

Case 2. d arbitrary. Let $J = (x_1, \dots, x_d)$ be a minimal reduction (see [NR]) of I . Then J is a one-fibered ideal as well since $\overline{\mathcal{R}(J)} = \overline{\mathcal{R}(I)}$. Suppose we have shown the theorem for J , with an integer $l' = l(R, J)$. Since J is a reduction of I , there is an integer t such that $I^{t+n} = J^nI^t$. If $f \notin I^n$, then $f \notin J^n$ and therefore there exist $g_2, \dots, g_d \in J$ such that $(f, g_2, \dots, g_d) \supseteq J^{n+l'}$. But $J^{n+l'} \supseteq I^{n+l'+t} = I^{n+l}$ which proves the theorem for I . Therefore we may assume $I = J$, i.e., I is generated by d elements. Denote by $S = R[x_2/x_1, \dots, x_d/x_1]$ the degree zero component of $\mathcal{R}(I)_{x_1}$, and let $T = \overline{S}$ be the integral closure of S . Furthermore, let Y_2, \dots, Y_d be indeterminates, denote by $u_i = x_1Y_i - x_i \in R[Y_2, \dots, Y_d]$, and denote by p the kernel of the R -epimorphism

$$\varphi: R[Y_2, \dots, Y_d] \rightarrow R[x_2/x_1, \dots, x_d/x_1], \quad Y_i \mapsto x_i/x_1.$$

Lemma 1.5. *There exists a $k \in \mathbb{N}$ such that*

$$I^k p \subseteq (u_2, \dots, u_d).$$

The proof of this lemma will be given after the conclusion of the proof of Theorem 1.4.

Since I is a one-fibered ideal $\text{rad}(IT) = Q$ is a prime ideal of T . Denote by $P = Q \cap S$. By the valuative criterion of integral dependence we get

$$I^r T_Q \cap R = \overline{I^r}.$$

Note that $\dim(S_P) = 1 = \dim(T_Q)$.

As R is analytically unramified there exists an $l_1 = l_1(R, I)$ such that $\overline{I^{n+l_1}} \subseteq I^n$ for all $n \geq 0$ ([R1, (1.4)]), and therefore if $f \notin I^n$ we have that $f \notin I^{n+l_1} S_P$. By a theorem of Rees ([R1, (1.5)]) T is a finite S -module. Since I is one-fibered, $T_Q = T_P (= T \otimes_S S_P)$, and therefore the integral closure of S_P , which is T_P , is local, and it is a finite S_P -module, implying that IS_P is one-fibered by Lemma 1.3 and that S_P is analytically unramified. Therefore we can apply the case $d = 1$ to IS_P , and we get that there exists an $l_2 = l_2(R, I)$ such that whenever $f \notin I^n S_P$ then $f S_P \supseteq I^{n+l_2} S_P$.

We get for $l_3 = l_1 + l_2$ that whenever $f \in R$, $f \notin I^n$ then $f S_P \supseteq I^{n+l_3} S_P$. Therefore there exists a $q \in S \setminus P$ such that

$$f S_q \supseteq I^{n+l_3} S_q,$$

i.e., such that

$$f(R[Y_2, \dots, Y_d]/p)_q \supseteq I^{n+l_3}(R[Y_2, \dots, Y_d]/p)_q,$$

implying that

$$(f, p)R[Y_2, \dots, Y_d]_{q(Y)} \supseteq I^{n+l_3}R[Y_2, \dots, Y_d]_{q(Y)},$$

where $q(Y)$ is a preimage of q in $R[Y_2, \dots, Y_d]$. Now let k be as in Lemma 1.5 and let $l = l_3 + k$. Then

$$(f, u_2, \dots, u_d)R[Y_2, \dots, Y_d]_{q(Y)} \supseteq I^{n+l}R[Y_2, \dots, Y_d]_{q(Y)}$$

by the lemma, with l depending on R and I only. Since $q(Y) \notin m(R) \times [Y_2, \dots, Y_d]$ and since R/m is infinite there exist $\alpha_2, \dots, \alpha_d \in R$ such that $q(\alpha_2, \dots, \alpha_d) \notin m$, and therefore $\alpha_2, \dots, \alpha_d$ induce an R -epimorphism

$$\psi: R[Y_2, \dots, Y_d]_{q(Y)} \rightarrow R,$$

and with $g_i := \psi(u_i)$ ($i = 2, \dots, d$) we get $g_i \in I$, and, by the above,

$$\begin{aligned} (f, g_2, \dots, g_d) &= \psi((f, u_2, \dots, u_d)R[Y_2, \dots, Y_d]_{q(Y)}) \\ &\supseteq \psi(I^{n+l}R[Y_2, \dots, Y_d]_{q(Y)}) \\ &= I^{n+l}. \quad \square \end{aligned}$$

Proof of Lemma 1.5. The map

$$\varphi: R[Y_2, \dots, Y_d] \rightarrow R[x_2/x_1, \dots, x_d/x_1], \quad Y_i \mapsto x_i/x_1,$$

arises from the homomorphism

$$\phi: R[Z_1, \dots, Z_d] \rightarrow R[x_1 T, \dots, x_d T], \quad Z_i \mapsto x_i T$$

by localizing in Z_1 , resp. x_1T , taking degree zero components and identifying Z_j/Z_1 with Y_j . Let P be the kernel of ϕ and denote by $a_{ij} = Z_i x_j - x_i Z_j \in P$. Furthermore denote by $A = (a_{ij})$ the homogeneous ideal generated by the elements a_{ij} . Denote by $(A_{Z_1})_0$ the degree zero component of the homogeneous ideal $AR[Z_1, \dots, Z_d]_{Z_1}$. Then $(A_{Z_1})_0 = (u_2, \dots, u_d)$ since $x_i Y_j - x_j Y_i = Y_i(x_1 Y_j - x_j Y_i) - Y_j(x_1 Y_i - x_i Y_1)$, and therefore it suffices to show

$$I^k P \subseteq A \quad \text{for some } k.$$

For this it obviously is enough to show that

$$x_i^n P \subseteq A \quad \text{for some } n \in \mathbb{N} \quad (i = 1, \dots, d)$$

or equivalently

$$PR[Z_1, \dots, Z_d]_{x_i} = AR[Z_1, \dots, Z_d]_{x_i}.$$

By symmetry we may assume $i = 1$. Then A_{x_1} is obviously generated by the elements $Z_i - x_i/x_1 Z_1$, and therefore $R[Z_1, \dots, Z_d]_{x_1}/A_{x_1} \cong R_{x_1}[Z_1]$, so that ϕ induces a homogeneous R -epimorphism

$$R[Z_1, \dots, Z_d]_{x_1}/A_{x_1} \cong R_{x_1}[Z_1] \rightarrow R[x_1T, \dots, x_dT]_{x_1} = R_{x_1}[T]$$

which therefore has to be an isomorphism, implying $P_{x_1} = A_{x_1}$ as desired. \square

Remark. We do not know whether the converse of Theorem 1.4 holds true. J. Sally showed us a proof of the converse if R is one-dimensional.

Theorem 1.6. *Let (R, m) be a complete local domain containing \mathbb{Q} , and let $I \subseteq R$ be a one-fibered m -primary ideal. Then there exists an $l = l(R, I)$ satisfying the following condition: If $f \in m$, $f \notin I^n$ then there exists a derivation $d \in \text{Der}(R)$ such that $d(f) \notin I^{n+l}$.*

The following easy consequence of the Artin-Rees Lemma will play a crucial role in the proof of Theorem 1.6:

Lemma 1.7. *Let R be a domain, let M be a finitely generated torsion-free R -module and let $I \subseteq R$ be an ideal of R . Then there exists an $l = l(M, I)$ such that, whenever $m \notin I^n M$, then there exists a $f \in \text{Hom}_R(M, R)$ with $f(m) \notin I^{n+l}$.*

Proof. Since R is a domain there exists a free R -module $F \subseteq M$ such that $F_r = M_r$ for some $r \in R \setminus \{0\}$. Since M is finitely generated we may assume that $rM \subseteq F$. Let f_1, \dots, f_d be a basis of F and denote by $\delta_1, \dots, \delta_d \in \text{Hom}_R(F, R)$ its dual basis. Then it clearly holds: If $x \notin I^n F$ then there exists an $i \in \{1, \dots, d\}$ such that $\delta_i(x) \notin I^n$.

By the Artin-Rees Lemma there exists an $l = l(M, I)$ such that, for all $n \in \mathbb{N}$,

$$I^{n+l} M \cap rM = I^n (rM \cap I^l M) \subseteq I^n rM \subseteq I^n F.$$

Since M is torsion-free, the map $\mu_r: M \rightarrow rM$ ($m \mapsto rm$) is an isomorphism, and therefore if $m \notin I^n M$ then $rm \notin I^n rM$, hence $rm \notin I^n (rM \cap I^l M) = I^{n+l} M \cap rM$. Since $rm \in rM$ this implies $rm \notin I^{n+l} M$, hence in particular

$rm \notin I^{n+l}F$. So by the above $\delta_i(rm) \notin I^{n+l}$ for some $i \in \{1, \dots, d\}$, and therefore the R -homomorphism f given by

$$f: M \xrightarrow{\mu_r} rM \hookrightarrow F \xrightarrow{\delta_i} R$$

will satisfy $f(m) \notin I^{n+l}$, proving the lemma. \square

Proof of Theorem 1.6. Denote by $L = Q(R)$ the quotient field of R , let $k \subseteq R$ be a coefficient field of R , and let $\tilde{\Omega}_{R/k}^1$ be the universally finite differential module of R/k (cf. [KD, §11]), and denote by $\tilde{\Omega}_{R/k}^*$ the module $\tilde{\Omega}_{R/k}^1/\text{torsion}$, which is the image of $\tilde{\Omega}_{R/k}^1$ by the canonical map

$$\tilde{\Omega}_{R/k}^1 \rightarrow (\tilde{\Omega}_{R/k}^1) \otimes_R L = D_k(L).$$

We have canonical isomorphisms

$$\text{Der}_k(R) \cong \text{Hom}_R(\tilde{\Omega}_{R/k}^1, R) \cong \text{Hom}_R(\tilde{\Omega}_{R/k}^*, R)$$

and therefore in view of Lemma 1.7 it suffices to show that there exists an $l_1 = l_1(R, I)$ such that the following holds: If $r \in m$, $r \notin I^n$ then $dr \notin I^{n+l_1}\tilde{\Omega}_{R/k}^*$, where by abuse of notation d denotes the composition

$$R \xrightarrow{d} \tilde{\Omega}_{R/k}^1 \xrightarrow{\text{can}} \tilde{\Omega}_{R/k}^*.$$

Let $t = \dim(R)$ and let $l_1 = l_1(R, I)$ be as in Theorem 1.4; i.e., if $r \notin I^n$ then there exist $g_2, \dots, g_t \in I$ such that $(r, g_2, \dots, g_t) \supseteq I^{n+l_1}$.

Since (r, g_2, \dots, g_t) is an m -primary ideal the inclusion

$$P = k[[r, g_2, \dots, g_t]] \hookrightarrow R$$

is a finite morphism. Note that P is a power series algebra in r, g_2, \dots, g_t . Let $K = Q(P)$ be the quotient field of P and let $D_k(K) = \tilde{\Omega}_{P/k}^1 \otimes_P K$. Then $D_k(L)$ is the universal L -extension of $D_k(K)$ by [KD, (11.9) and (4.21)], and therefore

$$D_k(L) = L \otimes_K D_k(K)$$

since L/K is finite and separable ([KD, (5.2)]). Hence the canonical trace ([KD, App. F])

$$\sigma_{L/K}: L \rightarrow K$$

induces a well defined $D_k(K)$ -linear map

$$\tilde{\sigma}_{L/K}: D_k(L) \rightarrow D_k(K)$$

which satisfies

$$\tilde{\sigma}_{L/K}(\tilde{\Omega}_{R/k}^*) \subseteq \tilde{\Omega}_{P/k}^1 = Pdr \oplus Pdg_2 \oplus \dots \oplus Pdg_t$$

as follows easily from [KW, (5.26)]. Hence we get

$$\tilde{\sigma}_{L/K}(dr) = dr \cdot \dim_K(L) \notin (r, g_2, \dots, g_t)\tilde{\Omega}_{P/k}^1$$

and therefore

$$dr \notin (r, g_2, \dots, g_t) \tilde{\Omega}_{R/k}^*$$

by the $D_k(K)$ -linearity of $\tilde{\sigma}_{L/K}$, implying

$$dr \notin I^{n+l_1} \tilde{\Omega}_{R/k}^*$$

as desired. \square

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