SOME PROPERTIES OF $k$-SEMISTRATIFIABLE SPACES

T. MIZOKAMI

(Communicated by Dennis Burke)

Abstract. We study spaces admitting semistratification and $k$-semistratifications with (CF) property. The class of $k$-semistratifiable spaces with (CF) property lies between the class of Lašnev spaces and that of $k$-semistratifiable spaces, and really differs from the classes of stratifiable spaces and N-spaces.

1. Introduction

All spaces are assumed to be regular Hausdorff topological spaces. The letter $\tau$ denotes the topology of a space $X$. We denote by the letter $\omega$ the set of all positive integers.

In his paper [8], Lutzer introduced the class of $k$-semistratifiable spaces, which lies between the class of stratifiable spaces in the sense of Borges [1] and Ceder [2] and the class of semistratifiable spaces introduced by Michael and studied by Creede. The class of $\sigma$-spaces introduced by Okuyama lies between that of stratifiable spaces and that of semistratifiable spaces. In this paper, we consider the limited classes of $k$-semistratifiable and semistratifiable spaces with (CF) property defined below. We give a few characterizations of Lašnev spaces in terms of $k$-semistratifiable spaces and CF families which are introduced here.

Throughout this paper, $\sigma$-spaces are spaces with a $\sigma$-discrete network or equivalently, $\sigma$-closure-preserving network, and N-spaces are spaces with a $\sigma$-locally finite $k$-network. Stratifiable spaces are spaces with the stratification. As for the definition of stratifications, refer to Borges [1].

2. $k$-SEMISTRATIFIABLE SPACES WITH (CF) PROPERTY

We state the original definition of $k$-semistratifiable spaces.

Definition 1 (Lutzer [8]). A space $X$ is called a $k$-semistratifiable space if there exists a function $S: \omega \times \tau \rightarrow \{\text{closed subsets of } X\}$ such that:

(a) For each $U \in \tau$, $U = \bigcup\{S(n, U) : n \in \omega\}$.
(b) If $U_1, V \in \tau$ and $U \subseteq V$, then $S(m, U) \subseteq S(m, V)$ for each $m$.
(c) If $C \subset U \in \tau$ with $C$ compact, then $C \subset S(m, U)$ for some $m$. (We call $S$ a $k$-semistratification of $X$.)

**Definition 2 ([10, Definition 3.1])**. A family $\mathcal{U}$ of subsets of a space $X$ is called finite on compact subsets of $X$, briefly CF in $X$, if $\mathcal{U}/K$ is a finite family for any compact subsets $K$ of $X$.

**Definition 3.** A semistratification or $k$-semistratification $S$ of a space $X$ is called to have (CF) property if the following condition (CF) is satisfied:

[(CF) For each $n \in \omega$, \{S\(n, U\): $U \in \tau$\} is CF in $X$.]

A space having $S$ with (CF) property is called a semistratifiable or a $k$-semistratifiable space with (CF) property, respectively.

**Theorem 1.** If a space $X$ has a $\sigma$-HCP (= hereditarily closure-preserving) $k$-network, then $X$ is a $k$-semistratifiable space with (CF) property.

**Proof.** Let $\bigcup\{\mathcal{H}_n: n \in \omega\}$ be a $k$-network for $X$, where, for each $n$, $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $\mathcal{H}_n$ is an HCP family of closed subsets of $X$. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup\{H \in \mathcal{H}_n: H \subset U\}.$$ 

Then it is easily seen from [10, Proposition 3.2] that $S$ is a $k$-semistratification with (CF) property.

**Example 1.** There exists a stratifiable, $k$-semistratifiable space with (CF) property, but does not have a $\sigma$-HCP $k$-network.

**Proof.** Let $Y$ be a non-metrizable Lašnev space which has no $\sigma$-locally finite $k$-network. (For example, let $Y$ be the quotient space obtained from $\bigoplus\{S_\alpha: \alpha < \omega_1\}$ by identifying all the limit points, where each $S_\alpha$ is the convergent sequence with its limit point.) Then by [6] the product space $X = Y \times [0, 1]$ has no $\sigma$-HCP $k$-network. $X$ is obviously a stratifiable space. By Theorem 5, stated later, $X$ is a $k$-semistratifiable space with (CF) property.

**Theorem 2.** For a space $X$, the following are equivalent:

1. $X$ is a Lašnev space.
2. $X$ is a Fréchet, $k$-semistratifiable space with (CF) property.
3. $X$ is a Fréchet space which has a $\sigma$-CF pseudobase.

**Proof.** (1) $\Rightarrow$ (2) follows from [3] and Theorem 1.

(2) $\Rightarrow$ (3). Let $S$ be the $k$-semistratification of $X$ with (CF) property. Then

$$\bigcup\{S(n, U): U \in \tau\}: n \in \omega\}$$

is a $\sigma$-CF pseudobase of $X$.

(3) $\Rightarrow$ (1) follows from [10, Theorem 4.1, (9)].

**Corollary.** A space $X$ is metrizable if and only if $X$ is a first countable, $k$-semistratifiable space with (CF) property.

We notice that a Lašnev space cannot be characterized to be a Fréchet space with a $\sigma$-HCP "pseudobase" [5]. For the next example, we prepare a lemma.
Lemma. If a space \( X \) has a \( \sigma \)-HCP \( k \)-network \( \mathcal{H} \) of closed subsets of \( X \), then \( X = X_1 \cup X_2 \), where \( X_1 \) is a \( \sigma \)-discrete closed subspace and \( X_2 \) is an \( \aleph \)-space such that for each \( p \in X_2 \), \( \mathcal{H} \) is \( \sigma \)-locally finite at \( p \) in \( X \).

Proof. Let \( \mathcal{H} = \bigcup \{ \mathcal{H}_n : n \in \omega \} \), where for each \( n \), \( \mathcal{H}_n \subset \mathcal{H}_{n+1} \) and \( \mathcal{H}_n \) is an HCP family of closed subsets of \( X \). Let

\[
X_1 = \{ p \in X : \bigcap \{ H \in \mathcal{H}_n : p \in H \} \text{ is a finite subset for some } n \}.
\]

Then by the same argument as in [11, Theorem 3.6], we can show that \( X_1 \) is a countable union of discrete closed subsets of \( X \) and \( X_2 = X - X_1 \) has the required property.

Example 2. There exists a stratifiable space which has no \( \sigma \)-HCP \( k \)-network.

Proof. For each \( \alpha < \omega_1 \), let \( T_\alpha \) be the copy of the subspace \( T = \{(x, y) : 0 < x, y \leq 1\} \) of \( \mathbb{R}^2 \) and \( f_\alpha : T \to T_\alpha \) its homeomorphism. Let \( X \) be the quotient space obtained from \( \bigoplus \{ T_\alpha : \alpha < \omega_1 \} \) by identifying \( \{ f_\alpha((x,0)) : \alpha < \omega_1 \} \) for each \( x \) with \( 0 \leq x \leq 1 \). Since \( X \) is dominated by metric spaces, \( X \) is a stratifiable space [1, Theorem 7.2]. If \( X \) has a \( \sigma \)-HCP \( k \)-network \( \mathcal{H} \), then by the above, there exists a point \( p = f(f_\alpha((x,0))) \in X \) such that \( \mathcal{H} \) is \( \sigma \)-locally finite at \( p \) in \( X \), where \( f : \bigoplus T_\alpha \to X \) is the quotient mapping. But, by [7, Remark 2] this is a contradiction.

Example 3. There exists a \( k \)-semistratifiable space which does not have \( \text{(CF)} \) property.

Proof. Let \( X \) be the same space as in [2, Example 9.2]. Then \( X \) is a first countable, non-metrizable stratifiable space. By the Corollary to Theorem 2, \( X \) has no \( k \)-semistratification with \( \text{(CF)} \) property.

From the argument as in Theorem 1, the following is easily seen.

Theorem 3. Any \( \sigma \)-space is a semistratifiable space with \( \text{(CF)} \) property.

The converse is not known. However, we have a partial answer to it.

Theorem 4. If a space \( X \) is a Fréchet, semistratifiable space with \( \text{(CF)} \) property, then \( X \) is a \( \sigma \)-space.

Proof. It is easy to see that \( X \) has a \( \sigma \)-CF network \( \mathcal{H} \). By [10, Proposition 3.3], \( \mathcal{H} \) is a \( \sigma \)-closure-preserving network. Hence \( X \) is a \( \sigma \)-space.

The following example shows that any semistratifiable space need not have \( \text{(CF)} \) property.

Example 4. There exists a first countable, semistratifiable space which is not a \( \sigma \)-space.

Proof. Let \( X \) be the space in [4, Example 9.10]. Since \( X \) is not a \( \sigma \)-space, by Theorem 4, \( X \) is not a semistratifiable space with \( \text{(CF)} \) property.
Theorem 5. If a space $X$ is embedded into a countable product of Lašnev spaces, then $X$ is a $k$-semistratifiable space with (CF) property.

Proof. By the same method as in [9, Lemma 5.1 and Proposition 6.1] and by [10, Proposition 3.3], we can show that $X$ has a $\sigma$-closure-preserving, CF family $\bigcup_n \mathcal{H}_n$ of closed subsets of $X$, which forms a $k$-network for $X$. For each $(n, U) \in \omega \times \tau$, let

$$S(n, U) = \bigcup \left\{ H \in \bigcup_{i \leq n} \mathcal{H}_i : H \subset U \right\}.$$  

Then $S$ is a $k$-semistratification with (CF) property.

The other known implications are indicated by the diagram below. The remaining proofs are easy and well-known, and therefore they are omitted.

References


Joetsu University of Education, Joetsu, Niigata 943, Japan