ON THE MODULUS
OF CONE ABSOLUTELY SUMMING OPERATORS
AND VECTOR MEASURES OF BOUNDED VARIATION

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(Communicated by John B. Conway)

Abstract. Let $E$ and $F$ be Banach lattices. It is shown that if $F$ has the
Levi and the Fatou property, then the ordered Banach space $\mathcal{L}^1(E, F)$ of
cone absolutely summing operators is a Banach lattice and an order ideal of
the Riesz space $\mathcal{L}^r(E, F)$ of regular operators. The same argument yields a
Jordan decomposition of $F$-valued vector measures of bounded variation.

Throughout the paper $E$ and $F$ denote Banach lattices and $\mathcal{F}$ an algebra
of subsets of some set $\Omega$. Identify $F$ with the canonical image of $F$ into its
bidual $F^{**}$. $F$ is said to have property (P), if there exists a positive contractive
projection $F^{**} \rightarrow F$. If $F$ has property (P), by Schlotterbeck's theorem [5]
the Banach space $\mathcal{L}^1(E, F)$ of cone absolutely summing operators is a Banach
lattice and an order ideal of the Riesz space $\mathcal{L}^r(E, F)$ of all regular operators
$E \rightarrow F$. Using this result Schmidt generalizes in [6] the Jordan decomposition
theorems of Diestel, Faires and Morrison [2, 3]. He proved that if $F$ has
property (P), then the ordered Banach space $\mathcal{bva}(\mathcal{F}, F)$ of all vector measures
$\mathcal{F} \rightarrow F$ having bounded variation is a Banach lattice and an order ideal of the
Riesz space $\mathcal{oba}(\mathcal{F}, F)$ of all order bounded vector measures $\mathcal{F} \rightarrow F$.

The main purpose of this note is to give a short proof of Schlotterbeck's
theorem. The proof is based on the notion of a Nakano space and avoids
duality arguments. The same type of proof yields Schmidt's theorem on vector
measures of bounded variation without representing them by cone absolutely
summing operators on the space of step functions.

Let us now fix some notions. A Banach lattice $F$ is said to satisfy the
Levi property (=weak Fatou property in [8]) if every increasing norm bounded
net of $F^+$ has a supremum. Note that the Levi property implies Dedekind
completeness. The norm of a Banach lattice $F$ is a Fatou norm if $0 \leq v, \forall v \in$
$F$ implies $||v_1|| \uparrow ||v||$. $F$ is called a Nakano space [1] if it satisfies the Levi property and if in addition its norm is Fatou.

Any dual Banach lattice is a Nakano space [1, Theorems 19.12 and 19.13] and has property (P) [4, p. 299]. Moreover, the class of Nakano spaces includes all Banach lattices having property (P). Indeed, since $F^{**}$ is a Nakano space, a norm bounded net $0 \leq v_\tau \uparrow$ of $F \subset F^{**}$ has a supremum $v^{**}$ in $F^{**}$ satisfying $||v^{**}|| = \sup ||v_\tau||$, and it follows easily that any positive contractive projection $P: F^{**} \rightarrow F$ satisfies $v_\tau \uparrow P v^{**} \in F$ and $||Pv^{**}|| = \sup ||v_\tau||$. No example seems to be known of a Nakano space which does not have property (P). As is well known, a Banach lattice $F$ with separating order continuous dual $F^*_n$ is a Nakano space if and only if it satisfies property (P), this holds if and only if $F$ is a perfect Banach lattice. Similarly, Dedekind complete $AM$-spaces with strong unit are the only $AM$-spaces having property (P) as well as the only Nakano $AM$-spaces.

We shall now state the theorem.

**Theorem 1.** Let $E, F$ be Banach lattices and suppose $F$ to be a Nakano space. Then $\mathcal{L}^l(E, F)$ is a Banach lattice and an order ideal of $\mathcal{L}^r(E, F)$.

**Proof.** Let $T \in \mathcal{L}^l(E, F)$, $u \in E^+$, and let $\pi(u)$ denote the collection of all finite families $\{u_1, \ldots, u_n\} \subset E^+$ satisfying $u = \sum u_i$. Since the set

$$D = \left\{ \sum_i |T u_i|: \{u_1, \ldots, u_n\} \in \pi(u) \right\}$$

is directed upward [8, p. 122] and norm bounded ($T$ is cone absolutely summing), by the Levi property there exists $y = \sup D$. According to [7, Lemma 2.1] $T \in \mathcal{L}^r(E, F)$ and $y = |T|u$. Moreover, by Fatou property we get

$$\sum_i ||T|u_i|| = \sum_i \left( \sup_{\pi(u)} \sum_j ||T u_{ij}|| \right) \leq \sum_i \left( \sup_{\pi(u)} \sum_j ||T u_{ij}|| \right) \leq \sup_{\pi(u)} \sum_j ||T u_{ij}|| \leq ||T||_l ||u||,$$

thus $|T| \in \mathcal{L}^l(E, F)$ and $|||T||_l \leq ||T||_r$. It follows that $\mathcal{L}^l(E, F)$ is an ideal of $\mathcal{L}^r(E, F)$, and since obviously $|||T||_l \leq ||T||_r$, the $l$-norm is a Riesz norm, and the proof is complete.

**Remarks.** 1. The proof combined with the duality principle [7, Lemma 2.2] and the duality of $l$- and $m$-norms [4, Theorem IV.3.8] yields a simplified proof of Schlotterbeck's theorem on the regularity of majorized operators [5].

2. The same type of proof is applicable to vector measures and yields the following Schmidt's decomposition theorem [6, Theorem 4.2].

**Theorem 2.** If $F$ is a Nakano space, then $\text{bva}(F, F)$ is a Banach lattice and an order ideal of $\text{oba}(F, F)$.
ACKNOWLEDGMENT

The author wishes to thank the referee for his helpful suggestions.

REFERENCES


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