DIRICHLET-FINITE ANALYTIC AND HARMONIC FUNCTIONS ARE BMO

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Abstract. Based on a result of F. John, an elementary proof is given of the fact that Dirichlet-finite analytic and Dirichlet-finite harmonic functions are of bounded mean oscillation in the unit disk.

1. In [5] Metzger proved the rather surprising result that the space of Dirichlet-finite analytic functions on a hyperbolic Riemann surface belong to the space BMO. Subsequently, in [4], Kusunoki and Taniguchi found that the same result holds for Dirichlet-finite harmonic functions on Riemann surfaces of finite type. In this note we give an elementary proof, based on a result of John [2], that in the unit disk Dirichlet-finite analytic and harmonic functions are BMO.

2. Denote the unit disk by \( U : |z| < 1 \), and

\[
AD(U) = \left\{ f \in A(U) : D_U(f) = \int_U \int |f'(z)|^2 \, dx \, dy < \infty \right\},
\]

\[
HD(U) = \left\{ u \in H(U) : D_U(u) = \int_U \int |\nabla u|^2 \, dx \, dy < \infty \right\},
\]

as the spaces of Dirichlet-finite analytic, and Dirichlet-finite harmonic, functions, respectively.

Let \( \text{BMO}_A(U) \) be the space of analytic functions on \( U \) which belong to the Hardy class \( H^2(U) \) and satisfy

\[
\sup_{z \in U} \int_U \int |f'(z)|^2 \log \left( 1 - \frac{z}{\zeta} \right) \, dx \, dy < \infty,
\]

and define \( \text{BMO}_H(U) \) analogously for harmonic functions, replacing \( |f'|^2 \) by \( |\nabla u|^2 \).

A sufficient condition for a real-valued differentiable function \( u \) to belong to \( \text{BMO}(U) \) in the sense of John–Nirenberg [3] has been given by John [2], namely:

\[
\sup_{z \in U} (1 - |z|) |\nabla u(z)| < \infty.
\]

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We can now readily establish the following:

**Theorem.** \( AD(U) \subseteq \text{BMOA}(U) \).

**Proof.** Firstly, \( AD(U) \subseteq H^2(U) \) (cf. Heins [1]). Take \( f \in AD(U) \), \( f = u + iv \). By the areal mean value property applied to \( \text{Re}[(f')^2] \), \( \text{Im}[(f')^2] \),

\[
(f'(z))^2 = \frac{1}{\pi \rho^2} \int_0^\rho \int_0^{2\pi} (f'(z + re^{i\theta}))^2 r \, dr \, d\theta,
\]

where \( \rho = 1 - |z| \). Then

\[
|\text{grad} \, u(z)|^2 = |f'(z)|^2 \leq \frac{1}{\pi(1 - |z|)^2} \int_U \int |f'(z)|^2 \, dx \, dy = \frac{1}{\pi(1 - |z|)^2} D_U(f).
\]

Therefore,

\[
\sup_{z \in U} (1 - |z|)|\text{grad} \, u(z)| \leq \sqrt{\frac{1}{\pi} D_U(f)} < \infty.
\]

Since \( u \in h^2(U) \), the John result implies \( u \in \text{BMOH}(U) \), and hence \( f \in \text{BMOA}(U) \).

For any function \( u \in HD(U) \), \( u \in \text{Re} \, f \) for some \( f \in AD(U) \), and the preceding proof yields:

**Corollary.** \( HD(U) \subseteq \text{BMOH}(U) \).

**References**