DIRICHLET-FINITE ANALYTIC
AND HARMONIC FUNCTIONS ARE BMO

J. L. SCHIFF

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ABSTRACT. Based on a result of F. John, an elementary proof is given of the
fact that Dirichlet-finite analytic and Dirichlet-finite harmonic functions are of
bounded mean oscillation in the unit disk.

1. In [5] Metzger proved the rather surprising result that the space of Dirichlet-
finite analytic functions on a hyperbolic Riemann surface belong to the space
BMO. Subsequently, in [4], Kusunoki and Taniguchi found that the same result
holds for Dirichlet-finite harmonic functions on Riemann surfaces of finite type.
In this note we give an elementary proof, based on a result of John [2], that in
the unit disk Dirichlet-finite analytic and harmonic functions are BMO.

2. Denote the unit disk by $U : |z| < 1$, and

$$AD(U) = \left\{ f \in A(U) : D_v(f) = \int_U \int |f'(z)|^2 \, dx \, dy < \infty \right\},$$

$$HD(U) = \left\{ u \in H(U) : D_v(u) = \int_U \int |\nabla u|^2 \, dx \, dy < \infty \right\},$$

as the spaces of Dirichlet-finite analytic, and Dirichlet-finite harmonic, func-
tions, respectively.

Let $BMOA(U)$ be the space of analytic functions on $U$ which belong to the
Hardy class $H^2(U)$ and satisfy

$$\sup_{z \in U} \int_U \int |f'(z)|^2 \log \left| \frac{1 - \zeta z}{z - \zeta} \right| \, dx \, dy < \infty,$$

and define $BMOH(U)$ analogously for harmonic functions, replacing $|f'|^2$ by
$|\nabla u|^2$.

A sufficient condition for a real-valued differentiable function $u$ to belong
to $BMO(U)$ in the sense of John–Nirenberg [3] has been given by John [2],
namely: $\sup_{z \in U} (1 - |z|)|\nabla u(z)| < \infty$.
We can now readily establish the following:

**Theorem.** \( AD(U) \subseteq BMOA(U) \).

**Proof.** Firstly, \( AD(U) \subseteq H^2(U) \) (cf. Heins [1]). Take \( f \in AD(U) \), \( f = u + iv \). By the areal mean value property applied to \( \text{Re}[(f')^2], \text{Im}[(f')^2] \),

\[
(f'(z))^2 = \frac{1}{\pi \rho^2} \int_0^\rho \int_0^{2\pi} [f'(z + re^{i\theta})]^2 r \, dr \, d\theta,
\]

where \( \rho = 1 - |z| \). Then

\[
|\nabla u(z)|^2 = |f'(z)|^2 \leq \frac{1}{\pi(1-|z|)^2} \int_U \int |f'(z)|^2 \, dx \, dy = \frac{1}{\pi(1-|z|)^2} D_U(f).
\]

Therefore,

\[
\sup_{z \in U} (1 - |z|)|\nabla u(z)| \leq \sqrt{\frac{1}{\pi} D_U(f)} < \infty.
\]

Since \( u \in h^2(U) \), the John result implies \( u \in BMOH(U) \), and hence \( f \in BMOA(U) \).

For any function \( u \in HD(U) \), \( u \in \text{Re} f \) for some \( f \in AD(U) \), and the preceding proof yields:

**Corollary.** \( HD(U) \subseteq BMOH(U) \).

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