ALMOST EVERYWHERE CONVERGENCE OF VILENKIN-FOURIER SERIES OF $H^1$ FUNCTIONS

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Abstract. In [5] Ladhawala and Pankratz proved that if $f$ is in dyadic $H^1$, then any lacunary sequence of partial sums of the Walsh-Fourier series of $f$ converges a.e. We generalize their theorem to Vilenkin-Fourier series. In obtaining this result, we prove a vector-valued inequality for the partial sums of Vilenkin-Fourier series.

1. Introduction

Let $G = \prod_{i=0}^{\infty} \mathbb{Z}_{p_i}$ be the countable direct product of cyclic groups of order $p_i$, where $\{p_i\}_{i \geq 0}$ is a sequence of integers with $p_i \geq 2$, and $\mu$ be the Haar measure on $G$ normalized by $\mu(G) = 1$. $G$ can be identified with the unit interval $(0,1)$ in the following manner. Set $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We associate with each $\{x_i\} \in G$, $0 < x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_i^{-1} \in (0,1)$. If we disregard the countable set of $p_i$-rationals, this mapping is one-one, onto, and measure-preserving.

For $x = \{x_i\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, \ldots$. We consider all finite products $\{\chi_n\}$ of $\{\phi_k\}$, enumerated according to a scheme of Paley. We express each nonnegative integer $n$ as a finite sum $n = \sum_{k=0}^{\infty} \alpha_k m_k$, with $0 \leq \alpha_k < p_k$, and define $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of $G$, and they form a complete orthonormal system on $G$. For the case $p_i = 2$, $i = 0, 1, \ldots$, $G$ is the dyadic group, $\{\phi_k\}$ are the Rademacher functions, and $\{\chi_n\}$ are the Walsh functions. In general, the system $(G, \{\chi_n\})$ is a realization of the Vilenkin systems studied in [7].

We consider the Fourier series with respect to $\{\chi_n\}$. For $f \in L^1(G)$, let

$$S_n f(x) = \int_G f(t) \sum_{j=0}^{n-1} \chi_j(x-t) \, d\mu(t), \quad n = 1, 2, \ldots,$$
be the $n$th partial sum of the Vilenkin-Fourier series of $f$. We define $H^1$ in terms of the $m_k$th partial sums $S_{m_k}f$, which are special for the Vilenkin-Fourier series. Let $\{G_k\}$ be a sequence of subgroups of $G$ defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \quad k = 1, 2, \ldots,$$

and $\mathcal{F}_k$ be the $\sigma$-algebra generated by the cosets of $G_k$. On the interval $(0,1)$, atoms of $\mathcal{F}_k$ are intervals of the form $(jm_k^{-1}, (j+1)m_k^{-1})$, $j = 0, 1, \ldots, m_k - 1$. We note that $\{\mathcal{F}_k\}$ is an increasing sequence of $\sigma$-algebras. Since $S_{m_k}f$ is the average of $f$ over the cosets of $G_k$ (see, e.g. [8, p. 312]), $\{S_{m_k}f, \mathcal{F}_k\}$ is a martingale.

For $f \in L^1(G)$, let $f^* = \sup_{k\geq 0} \|S_{m_k}f\|$ and

$$S(f) = \left[ \sum_{k=-1}^{\infty} (S_{m_{k+1}}f - S_{m_k}f)^2 \right]^{1/2}, \quad \text{where } S_{m_{-1}}f = 0.$$

Applying Davis' result for martingales [1], we know that there exist positive constants $c$ and $C$ (independent of the orders $\{p_i\}$) such that

$$(1.1) \quad c\|S(f)\|_1 \leq \|f^*\|_1 \leq C\|S(f)\|_1.$$  

We say that $f \in H^1(G)$ if $S(f) \in L^1(G)$, or, equivalently, $f^* \in L^1(G)$, and we write

$$\|f\|_{H^1} = \|S(f)\|_1.$$  

Our definition of $H^1(G)$ is a special case of the definition of $H_1$ for martingales given by Garsia [3]. For other definitions of $H^1(G)$, see [2] and [6].

We have the following theorem concerning the a.e. convergence of Vilenkin-Fourier series of functions in $H^1(G)$.

**Theorem 1.** Let $f \in H^1(G)$ and let $\{n_k\}_{k\geq 0}$ be a sequence of positive integers such that $m_k \leq n_k < m_{k+1}$, $k = 0, 1, \ldots$. Then, as $k \to \infty$, $S_{n_k}f(x) \to f(x)$ for a.e. $x \in G$.

For the case where $G$ is the dyadic group, the theorem is proved by Ladhawala and Pankratz [5]. If $f \in L^p(G)$, $1 < p < \infty$, then $f^* \in L^p(G)$ by Doob's inequality. Hence $L^p(G) \subset H^1(G)$. Even for the case where $f \in L^p(G)$, $1 < p < \infty$, our result is new if the orders of the cyclic groups are unbounded, i.e. $\sup_i p_i = \infty$. For the bounded case, $\sup_i p_i < \infty$, Gosselin [4] showed that if $f \in L^p(G)$, $1 < p < \infty$, the full sequence of partial sums $\{S_nf\}$ converges a.e. to $f$.

To prove Theorem 1, we shall show that, for any sequence $\{n_k\}$ with $m_k \leq n_k < m_{k+1}$, $k = 0, 1, \ldots$, we have

$$(1.2) \quad \mu \left\{ x \in G : \sup_{k\geq 0} |S_{n_k}f(x)| > y \right\} \leq Cy^{-1}\|f\|_{H^1},$$

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where \( y > 0 \), \( f \in H^1(G) \), and \( C \) is an absolute constant independent of the orders \( \{p_i\} \). Since \( S_{m_k} f \) converges to \( f \) in the \( H^1 \) norm, Theorem 1 will then follow by the usual density argument.

We shall obtain (1.2) as a consequence of a vector-valued inequality concerning the partial sums of Vilenkin-Fourier series.

**Theorem 2.** There exist constants \( C \) and \( C_p \) such that, for any sequence \( \{f_i\} \) of functions in \( L^1(G) \) and any sequence of positive integers \( \{n_i\} \),

\[
\mu \left\{ x \in G : \left( \sum_{l=0}^{\infty} |S_{n_l} f_l(x)|^2 \right)^{1/2} > y \right\} \leq Cy^{-1} \left\| \left( \sum_{l=0}^{\infty} |f_l|^2 \right)^{1/2} \right\|_1, \quad y > 0,
\]

\[
\left\| \left( \sum_{l=0}^{\infty} |S_{n_l} f_l|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left( \sum_{l=0}^{\infty} |f_l|^2 \right)^{1/2} \right\|_p, \quad 1 < p < \infty.
\]

The constants \( C \) and \( C_p \) are independent of the orders \( \{p_i\} \) of the cyclic groups.

In what follows, \( C \) will denote an absolute constant, which may vary from line to line.

### 2. A DECOMPOSITION LEMMA

To prove Theorem 2, we use a Calderón-Zygmund type decomposition lemma. This lemma is a modified version of the one given in [8; Lemma 2, p. 314]. We shall describe it on the interval \((0, 1)\).

**Lemma 3.** Let \( y > 0 \) and \( \{f_i\}_{i \geq 0} \) be a sequence of functions on \( G \) such that \( \left\| (\sum_i |f_i|^2)^{1/2} \right\|_1 \leq y \). Let \( \{\alpha_{i,k}\}_{i,k \geq 0} \) be a double sequence of integers with \( 0 \leq \alpha_{i,k} < p_k \). Then there are sequences of functions \( \{g_i\}_{i \geq 0}, \{b_i\}_{i \geq 0} \) on \( G \) and a collection \( \mathcal{E} = \{\omega_j\} \) of disjoint intervals such that

\[
f_i = g_l + b_l, \quad l = 0, 1, \ldots.
\]

\[
\left( \sum_l |g_l|^2 \right)^{1/2} \leq Cy \quad \text{a.e.}
\]

\[
\left\| \left( \sum_l |g_l|^2 \right)^{1/2} \right\|_1 \leq C \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|_1.
\]

\[
\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k,
\]

where each \( \omega_j \in \mathcal{E}_k \) is measurable with respect to \( \mathcal{F}_{k+1} \), and is a proper subset of a coset of \( G_k \).

\[
b_l(x) = 0 \quad \text{if} \quad x \notin \Omega \equiv \bigcup_j \omega_j, \quad l = 0, 1, \ldots.
\]

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For each \( l = 0, 1, \ldots \),
\[
\int_{\omega_j} b_l d\mu = 0 \quad \text{for every } \omega_j \in \mathcal{E}, \quad \text{and}
\]
(2.6)
\[
\int_{\omega_j} b_l \phi_k^{\alpha_k} d\mu = 0 \quad \text{for every } \omega_j \in \mathcal{E}, \quad k = 0, 1, \ldots .
\]
(2.7) \[
\int_{\omega_j} \left( \sum_l |b_l|^2 \right)^{1/2} d\mu \leq C \int_{\omega_j} \left( \sum_l |f_l|^2 \right)^{1/2} d\mu \quad \text{for every } \omega_j \in \mathcal{E}.
\]
(2.8) \[
\sum_j \mu(\omega_j) \leq y^{-1} \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|.
\]

Proof. We apply the decomposition in [8] (see proof of Lemma 2, pp. 314–315) to the function \( (\sum_l |f_l|^2)^{1/2} \) to obtain a collection \( \mathcal{E} = \{\omega_j\} \) of disjoint intervals with the properties that

(2.9) \[
y < \frac{1}{\mu(\omega_j)} \int_{\omega_j} \left( \sum_l |f_l|^2 \right)^{1/2} d\mu \leq 3y, \quad \omega_j \in \mathcal{E},
\]
(2.10) \[
\left( \sum_l |f_l(x)|^2 \right)^{1/2} \leq y, \quad \text{for a.e. } x \not\in \Omega,
\]
and that \( \mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k \), where \( \{\mathcal{E}_k\} \) satisfies (2.4). The first inequality of (2.9) then implies (2.8).

Next, we decompose \( f_l = g_l + b_l, \ l = 0, 1, \ldots \), with
\[
g_l(x) = \begin{cases} f_l(x) & \text{if } x \not\in \Omega, \\ a_{lkj} + b_{lkj} \phi_k^{-\alpha_k}(x) & \text{if } x \in \omega_j \in \mathcal{E}_k, 
\end{cases}
\]
where \( a_{lkj}, b_{lkj} \) are constants chosen in such a way that
\[
\int_{\omega_j} f_l d\mu = \int_{\omega_j} (a_{lkj} + b_{lkj} \phi_k^{-\alpha_k}) d\mu,
\]
and
\[
\int_{\omega_j} f_l \phi_k^{\alpha_k} d\mu = \int_{\omega_j} (a_{lkj} + b_{lkj} \phi_k^{-\alpha_k}) \phi_k^{\alpha_k} d\mu.
\]
Then \( b_l = g_l - f_l \) satisfies (2.5) and (2.6). Also, it follows from the proof of (25) in [8, pp. 315–317] that
\[
|g_l(x)| \leq \frac{C}{\mu(\omega_j)} \int_{\omega_j} |f_l| d\mu, \quad x \in \omega_j, \ \omega_j \in \mathcal{E}.
\]
Hence, by Minkowski's inequality for integrals, we have
\[
\left( \sum_l |g_l(x)|^2 \right)^{1/2} \leq \frac{C}{\mu(\omega_j)} \int_{\omega_j} \left( \sum_l |f_l|^2 \right)^{1/2} d\mu, \quad x \in \omega_j, \ \omega_j \in \mathcal{E}.
\]
This inequality, together with (2.9) and (2.10), implies (2.2), (2.3) and (2.7). \( \square \)
3. Proof of Theorem 2

We shall prove (1.3). The case \( p = 2 \) of (1.4) is a consequence of Plancherel’s formula. For \( 1 < p < 2 \), (1.4) follows from (1.3) and the case \( p = 2 \) by the vector-valued Marcinkiewicz interpolation theorem. The case \( 2 < p < \infty \) then follows by a duality argument.

Instead of proving (1.3), we shall prove an equivalent inequality involving the modified partial sums \( \{S_n^*f\} \). For \( f \in L^1(G) \), let

\[ S_n^*f = \bar{x}_n S_n(f \bar{x}_n), \quad n = 1, 2, \ldots \]

(For the properties of \( S_n^*f \), see [8, pp. 313–314].) (1.3) is equivalent to

\[
\frac{1}{2} \mathcal{E} \mathcal{X} x \in G: \left( \sum_{k=0}^{\infty} |S_n^*f_k(x)|^2 \right)^{1/2} > y \leq C y^{-1} \left( \sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2},
\]

where \( y > 0 \), \( \{f_k\} \) is any sequence of functions in \( L^1(G) \), and \( \{n_k\} \) is any sequence of positive integers.

To prove (3.1), we can assume \( \| (\sum_l |f_l|^2)^{1/2} \|_1 \leq y \). For each \( l = 0, 1, \ldots \), decompose \( f_l \) as in Lemma 3. Since

\[
\mu \left\{ \left( \sum_l |S_n^*f_l|^2 \right)^{1/2} > y \right\} \leq \mu \left\{ \left( \sum_l |S_n^*g_l|^2 \right)^{1/2} > y/2 \right\} + \mu \left\{ \left( \sum_l |S_n^*b_l|^2 \right)^{1/2} > y/2 \right\},
\]

(3.1) will be proved if we can show that each term on the right is bounded by \( C y^{-1} \| (\sum_l |f_l|^2)^{1/2} \|_1 \).

Using Plancherel’s formula, we obtain

\[
\mu \left\{ \left( \sum_l |S_n^*g_l|^2 \right)^{1/2} > y/2 \right\} \leq C y^{-2} \left\| \left( \sum_l |S_n^*g_l|^2 \right)^{1/2} \right\|_2^2 \leq C y^{-2} \left\| \left( \sum_l |g_l|^2 \right)^{1/2} \right\|_2^2 \leq C y^{-1} \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|_1^2,
\]

by (2.2) and (2.3).

To estimate the second term in (3.2), we use the following notation. Let \( \omega_j \in F_{k+1} \), with \( \omega_j \) contained in the coset \( I \) of \( G_k \). We consider \( I \) as a circle and let \( \omega_j^* \) denote the interval inside \( I \) which contains \( \omega_j \) at its center.
with $\mu(\omega_j^*) = 3\mu(\omega_j)$. Let $\Omega^* = \bigcup_j \omega_j^*$. Then, by (2.8),

$$\mu(\Omega^*) \leq 3 \sum_j \mu(\omega_j) \leq 3y^{-1} \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|_1.$$

Therefore, it suffices to prove

(3.3) $\mu \left\{ x \notin \Omega^* : \left( \sum_l |S_{n_l} b_l(x)|^2 \right)^{1/2} > y/2 \right\} \leq C y^{-1} \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|_1.$$

To do this, we express $S_{n_l} b_l$ in terms of the conjugate functions. Let $f \in L^1(G)$. For $x \in \{x_k\} \in G$, $I = x + G_k$, define

$$H_k f(x) = \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq \text{int}_k\}} f(t) \cot(\pi(x_k - t_k)/p_k) \, d\mu(t).$$

If $n_i = \sum_{k=0}^\infty \alpha_{ik} m_k$, $0 \leq \alpha_{ik} < p_k$, it is shown in [8, pp. 313–314] that

$$S_{n_l} b_l(x) = \sum_{k=0}^\infty \frac{\alpha_{ik}}{\mu(I)} \int_{I \cap \{x_k = \text{int}_k\}} b_l(t) \, d\mu(t)$$

$$+ \frac{1}{2} \sum_{k=0}^\infty \phi_k^{-\alpha_{ik}}(x) \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq \text{int}_k\}} b_l(t) \phi_k^{\alpha_{ik}}(t) \, d\mu(t)$$

$$- \frac{1}{2} \sum_{k=0}^\infty \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq \text{int}_k\}} b_l(t) \, d\mu(t)$$

$$+ i \sum_{k=0}^\infty \phi_k^{-\alpha_{ik}}(x) H_k(b_l \phi_k^{\alpha_{ik}})(x)$$

$$- i \sum_{k=0}^\infty H_k b_l(x).$$

For $x \notin \Omega^*$, (2.5) and (2.6) imply that the first three terms on the right vanish. (See the explanation below (29) on p. 317 in [8].) Thus, it follows from Minkowski's inequality that, for $x \notin \Omega^*$,

$$\left[ \sum_l |S_{n_l} b_l(x)|^2 \right]^{1/2} \leq \sum_{k=0}^\infty \left[ \sum_l |H_k(b_l \phi_k^{\alpha_{ik}})(x)|^2 \right]^{1/2}$$

$$+ \sum_{k=0}^\infty \left[ \sum_l |H_k b_l(x)|^2 \right]^{1/2}.$$

(3.3) will be proved if we show

(3.4) $\mu \left\{ x \notin \Omega^* : \sum_{k=0}^\infty \left[ \sum_l |H_k(b_l \phi_k^{\alpha_{ik}})(x)|^2 \right]^{1/2} > y/4 \right\}$

$$\leq C y^{-1} \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|_1.$$

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and

\[
\mu \left\{ x \notin \Omega^* : \sum_{k=0}^{\infty} \left[ \sum_{l} |H_k b_l(x)|^2 \right]^{1/2} > y/4 \right\} \\
\leq C y^{-1} \left\| \left( \sum_{l} |f_l|^2 \right)^{1/2} \right\|_1.
\]

We shall demonstrate (3.4). (3.5) can be proved similarly.

Suppose \( x \notin \Omega^* \) and \( I = x + G_k \). Again it follows from (2.5) and (2.6) that

\[
H_k(b, \Omega^*) = \sum_{\omega \subset I : \omega \in \mathcal{E}_k} \int_{\omega} b_l(t) \phi_k^{\alpha_l}(t) \times \left[ \cot \left( \frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left( \frac{\pi(x_k - t'_k)}{p_k} \right) \right] d\mu(t),
\]

where \( t' = \{t'_l\}_{l \geq 0} \) is any fixed point in \( \omega_j \). (See the proof below (32) on p. 138 in [8].) By Minkowski's inequality for integrals, we have

\[
\left[ \sum_l |H_k(b, \Omega^*)(x)|^2 \right]^{1/2} \leq \frac{1}{2} \frac{1}{\mu(I)} \sum_{\omega \subset I : \omega \in \mathcal{E}_k} \int_{\omega} \left[ \sum_l |b_l(t)|^2 \right]^{1/2} \\
\times \left| \cot \left( \frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left( \frac{\pi(x_k - t'_k)}{p_k} \right) \right| d\mu(t).
\]

Thus, for any coset \( I \) of \( G_k \), Fubini's theorem gives

\[
\int_{I \cap \Omega^*} \left[ \sum_l |H_k(b, \Omega^*)(x)|^2 \right]^{1/2} d\mu(x) \\
\leq \frac{1}{2} \frac{1}{\mu(I)} \sum_{\omega \subset I : \omega \in \mathcal{E}_k} \int_{\omega} \left[ \sum_l |b_l(t)|^2 \right]^{1/2} \\
\times \int_{I \cap \omega_j^*} \left| \cot \left( \frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left( \frac{\pi(x_k - t'_k)}{p_k} \right) \right| d\mu(x) d\mu(t).
\]

Since, for \( t \in \omega_j \), a simple computation gives

\[
\frac{1}{\mu(I)} \int_{I \cap \omega_j^*} \left| \cot \left( \frac{\pi(x_k - t_k)}{p_k} \right) - \cot \left( \frac{\pi(x_k - t'_k)}{p_k} \right) \right| d\mu(x) d\mu(t) \leq C,
\]

we have

\[
\int_{I \cap \Omega^*} \left[ \sum_l |H_k(b, \Omega^*)(x)|^2 \right]^{1/2} d\mu \leq C \sum_{\omega \subset I : \omega \in \mathcal{E}_k} \int_{\omega} \left( \sum_l |b_l|^2 \right)^{1/2} d\mu \\
\leq C \sum_{\omega \subset I : \omega \in \mathcal{E}_k} \int_{\omega} \left( \sum_l |f_l|^2 \right)^{1/2} d\mu,
\]

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by (2.7). Therefore

\[
\mu \left\{ x \notin \Omega^* : \sum_{k=0}^{\infty} \left[ \sum_l |H_k(b_l\phi_k^0(x))|^2 \right]^{1/2} > y/4 \right\}
\leq C y^{-1} \sum_{k=0}^{\infty} \int_{\Omega^*} \left[ \sum_l |H_k(b_l\phi_k^0(x))|^2 \right]^{1/2} d\mu
\leq C y^{-1} \sum_{k=0}^{\infty} \sum_{\omega_j \in \Phi_k} \left( \sum_l |f_l|^2 \right)^{1/2} d\mu
\leq C y^{-1} \left\| \left( \sum_l |f_l|^2 \right)^{1/2} \right\|_1.
\]

This establishes (3.4), and hence completes the proof of Theorem 2.

4. Proof of Theorem 1

It suffices to prove (1.2). Let \( f \in H^1(G) \) and \( f_k = S_{m_k+1}f - S_{m_k}f, \ k = 0, 1, \ldots \). For \( m_k \leq n_k < m_{k+1} \), \( S_{n_k}f = S_{m_k}f + S_{n_k}f_k \). Hence, for \( y > 0 \),

\[
\mu \left\{ \sup_{k \geq 0} |S_{n_k}f| > y \right\} \leq \mu \left\{ \sup_{k \geq 0} |S_{m_k}f| > y/2 \right\} + \mu \left\{ \sup_{k \geq 0} |S_{n_k}f_k| > y/2 \right\}.
\]

By (1.1),

\[
\mu \left\{ \sup_{k \geq 0} |S_{m_k}f| > y/2 \right\} \leq 2y^{-1} \|f\|_{H^1} \leq Cy^{-1} \|f\|_{H^1}.
\]

From (1.3) of Theorem 2,

\[
\mu \left\{ \sup_{k \geq 0} |S_{n_k}f_k| > y/2 \right\} \leq \mu \left\{ \left( \sum_{k=0}^{\infty} |S_{n_k}f_k|^2 \right)^{1/2} > y/2 \right\}
\leq C y^{-1} \left\| \left( \sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2} \right\|_1 \leq C y^{-1} \|f\|_{H^1}.
\]

This completes the proof of Theorem 1.

References


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