STARSHAPED UNIONS AND NONEMPTY INTERSECTIONS OF CONVEX SETS IN $R^d$

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Abstract. Let $\mathcal{F}$ be a nonempty family of compact convex sets in $R^d$, $d \geq 1$. Then every subfamily of $\mathcal{F}$ consisting of $d+1$ or fewer sets has a starshaped union if and only if $\cap\{G: G \in \mathcal{F}\} \neq \emptyset$.

1. Introduction

We begin with some definitions. Let $S$ be a subset of $R^d$. For points $x$ and $y$ in $S$, we say $x$ sees $y$ via $S$ if and only if the corresponding segment $[x, y]$ lies in $S$. Set $S$ is called starshaped if and only if there is some point $p$ in $S$ such that $p$ sees via $S$ each point of $S$, and the set of all such points $p$ is the (convex) kernel of $S$.

A familiar theorem by Krasnosel'skii [4] states that for $S$ a nonempty compact set in $R^d$, $S$ is starshaped if and only if every $d+1$ points of $S$ see via $S$ a common point. In studying starshaped unions of sets, Kolodziejczyk [3] has proved that for $\mathcal{F}$ a finite family of closed sets in $R^d$, if every $d+1$ members of $\mathcal{F}$ have a starshaped union, then $\cup\{F: F \in \mathcal{F}\}$ is starshaped as well. In this paper, we examine the relationship between starshaped unions and nonempty intersections of compact convex sets in $R^d$ to obtain the following Helly-type analogue: Let $\mathcal{F}$ be a nonempty family of compact convex sets in $R^d$, $d \geq 1$. Then every subfamily of $\mathcal{F}$ consisting of $d+1$ or fewer sets has a starshaped union if and only if $\cap\{G: G \in \mathcal{F}\} \neq \emptyset$. (Of course, when members of $\mathcal{F}$ have a nonempty intersection, they will have a starshaped union as well.)

The proof is suggested by an argument of Klee [2].

Throughout the paper, $\text{conv} S$, $\text{int} S$, $\text{bdry} S$, and $\text{ker} S$ will denote the convex hull, interior, boundary, and kernel, respectively, for set $S$. For distinct points $x$ and $y$, $L(x, y)$ will be the line they determine. The reader is referred...

The results

The following definition is needed.

**Definition.** Set $A$ is said to surround set $B$ in the $k$-flat $F$, $k \geq 1$, if and only if $A$ contains a $(k - 1)$-sphere $S$ such that $B$ lies in the bounded component of $F \sim S$.

Our preliminary lemma is motivated by an argument of Klee [2].

**Lemma 1.** Let $K_1, \ldots, K_l$ be nonempty compact convex sets in $\mathbb{R}^d$, $d \geq 1$, $l \geq 2$, with $\bigcap\{K_i: 1 \leq i \leq l\} = \emptyset$ and with $a_i \in \bigcap\{K_j: 1 \leq j \leq l, j \neq i\} \neq \emptyset$ for $1 \leq i \leq l$. Then there are two flats $H$ and $L$ of dimension $l - 1$ and $d - l + 1$, respectively, meeting in a single point, such that

1. $L \cap K_i = \emptyset$ and $a_i \in H$, $1 \leq i \leq l$, and
2. $H \cap (\bigcup\{K_i: 1 \leq i \leq l\})$ surrounds $H \cap L$ in $H$.

**Proof.** Clearly Helly's familiar theorem, together with the hypothesis of the lemma, imply that $2 \leq l \leq d + 1$. We proceed by induction on $d$. If $d = 1$, then $l = 2$, and it is easy to see that the lemma holds. For $d > 1$, assume the result is true for integers $k$, $1 \leq k \leq d$, to prove for $d$. Since $\bigcap\{K_i: 1 \leq i \leq l\} = \emptyset$, let $H_0$ be a hyperplane strictly separating the compact convex sets $K_i$ and $\bigcap\{K_j: 2 \leq j \leq l\} \neq \emptyset$.

In case $l = 2$, let $H = L(a_1, a_2)$ and let $L = H_0$. If $l \geq 3$, choose $\{a'_i\} = [a_1, a_l] \cap H_0$, $2 \leq i \leq l$. Since $a'_i \in \bigcap\{K_j: j \neq 1, i\}$, every $l - 2$ sets from $\{K_l \cap H_0: 2 \leq i \leq l\}$ have a nonempty intersection. However, $H_0$ is disjoint from $\bigcap\{K_j: 2 \leq j \leq l\}$, so $\bigcap\{K_l \cap H_0: 2 \leq i \leq l\} = \emptyset$. Using our induction hypothesis in the $(d - 1)$-flat $H_0$, there exist flats $H'$, $L$ in $H_0$ having dimension $(l - 1) - 1 = l - 2$ and $(d - 1) - (l - 1) + 1 = d - l + 1$, respectively, meeting in a single point, such that

1. $L \cap K_i = \emptyset$ and $a'_i \in H'$, $2 \leq i \leq l$, and
2. $H' \cap (\bigcup\{K_i: 2 \leq i \leq l\})$ surrounds $H' \cap L$ in $H'$.

Finally, let $H$ be the flat determined by $H'$ and $a_1$. Clearly $a_i \in L(a_1, a'_l) \subseteq H$ for $2 \leq i \leq l$, and hence $a_i \in H$, $1 \leq i \leq l$. Moreover, since $\text{bdry conv}\{a_1, \ldots, a_l\} \subset H \cap (\bigcup\{K_i: 1 \leq i \leq l\})$, $H \cap (\bigcup\{K_i: 1 \leq i \leq l\})$ surrounds $H \cap L = H \cap L$ in $H$. This finishes the induction and completes the proof of the lemma.

**Theorem.** Let $\mathcal{F}$ be a nonempty family of compact convex sets in $\mathbb{R}^d$, $d \geq 1$. Then every subfamily of $\mathcal{F}$ consisting of $d + 1$ or fewer sets has a starshaped union if and only if $\bigcap\{G: G \text{ in } \mathcal{F}\} \neq \emptyset$.

**Proof.** Clearly when $\bigcap\{G: G \text{ in } \mathcal{F}\} \neq \emptyset$, then every subfamily of $\mathcal{F}$ has a starshaped union whose kernel contains $\bigcap\{G: G \text{ in } \mathcal{F}\}$. Hence we need only establish the reverse implication.
Assume that every \( d + 1 \) or fewer sets in \( \mathcal{G} \) have a starshaped union, to show that \( \cap\{G: G \text{ in } \mathcal{G}\} \neq \emptyset \). Note that for arbitrary sets \( G_1 \) and \( G_2 \) in \( \mathcal{G} \), \( G_1 \cup G_2 \) is starshaped. Since both \( G_1 \) and \( G_2 \) are closed, this implies that \( G_1 \cap G_2 \neq \emptyset \), and thus every two members of \( \mathcal{G} \) intersect. By the familiar Helly theorem, it suffices to prove that every \( d + 1 \) or fewer members of \( \mathcal{G} \) have a nonempty intersection, \( 2 \leq d \).

Suppose on the contrary that for some maximal integer \( l - 1 \), \( 2 \leq l - 1 \leq d \), every \( l - 1 \) members of \( \mathcal{G} \) have a nonempty intersection but some \( l \) members of \( \mathcal{G} \) have an empty intersection. Say \( G_i \cap \cdots \cap G_l = \emptyset \) for \( G_i \) in \( \mathcal{G} \), \( 1 \leq i \leq l \).

By Lemma 1, there exist flats \( H, L \) of dimension \( l - 1, d - l + 1 \), respectively, meeting in a single point, such that

1. \( L \cap G_i = \emptyset, \ 1 \leq i \leq l \), and
2. \( H \cap (\cup\{G_i: 1 \leq i \leq l\}) \) surrounds \( H \cap L \) in \( H \).

However, this contradicts the fact that \( \cup\{G_i: 1 \leq i \leq l\} \) is starshaped. Our supposition is false, and \( \cap\{G: G \text{ in } \mathcal{G}\} \neq \emptyset \), finishing the proof of the theorem.

Remark. It is interesting to observe that Theorem 1 holds without the requirement that members of \( \mathcal{G} \) be compact, provided \( \mathcal{G} \) is a finite family whose members are closed: In the proof, simply choose \( x \in \ker(\cup\{G_i: 1 \leq i \leq l\}) \neq \emptyset, \ a_i \in \cap\{G_j: 1 \leq j \leq l \ j \neq i\} \), and define \( T \equiv \text{conv}\{x, a_i: 1 \leq i \leq l\} \). Then apply Lemma 1 to \( \{T \cap G_i: 1 \leq i \leq l\} \). The finite version of Helly’s theorem completes the argument.

However, the theorem fails without the restriction that members of \( \mathcal{G} \) be closed, as the following easy example illustrates.

Example 1. Let \( s_1, \ldots, s_{d+1} \) be vertices of a \( d \)-simplex in \( R^d \), with \( w \in \text{int conv}\{s_1, \ldots, s_{d+1}\} \). For \( 1 \leq i \leq d + 1 \), define

\[
S_i = \text{conv}\{w, s_j: 1 \leq j \leq d + 1, j \neq i\}
\]

and let \( T_i = S_i \sim \{w\} \). Every \( d \) (or fewer) of the sets \( T_1, S_2, \ldots, S_{d+1} \) intersect and hence have a starshaped union. Furthermore, \( T_1 \cup S_2 \cup \cdots \cup S_{d+1} = \text{conv}\{s_1, \ldots, s_{d+1}\} \) is convex and hence starshaped. However, \( T_1 \cap S_2 \cap \cdots \cap S_{d+1} = \emptyset \).

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References


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