SEQUENCES OF REALS
TO SEQUENCES OF ZEROS AND ONES

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ABSTRACT. This paper considers maps from sequences of reals to sequences of zeros and ones which preserve the tail. The continuum hypothesis is used.

0. INTRODUCTION

In this paper $\mathbf{N}$ does not include 0.

Let $A = \mathscr{R}^{\mathbf{N}}$ and $B = \{0,1\}^{\mathbf{N}}$. The purpose of this paper is to discuss certain functions $F\colon A \to B$. A function $F\colon A \to B$ is said to be good if:

(1) for $\{a_i\} \in A$, $\{b_i\} \in A$, (there exists N such that $n > N \to a_n = b_n$) iff there exists M such that $m > M \to (F(\{a_i\}))_m = (F(\{b_i\}))_m$;

(2) $(F(\{a_i\}))_m$ depends only on a_1, a_2, ..., a_m.

A function $F\colon A \to B$ is said to be m-good if:

(a) condition 1 above;

(b) F is measurable (Lebesgue measure and σ-algebras).

The purpose of this paper is to show the continuum hypothesis implies the existence of good maps (see the note below); and m-good maps do not exist. The first result is much deeper than the second.

Question: Does the converse of 1 hold?

Note. We prove (1) by proving the existence of a good map from $\aleph_1^{\mathbf{N}} \to \{0,1\}^{\mathbf{N}}$ (we do not need continuum hypothesis for this). Then, we get (1) by assuming $\aleph_1 = C$.

1. m-GOOD MAPS DO NOT EXIST

Elements of $\mathscr{R}^{\mathbf{N}}$ can be considered as sequences of reals, where r_n is the nth real, and each r_n can be considered to be a sequence of zeros and ones where $r_{n,j}$ is the jth entry of r_n. In this manner, elements of $\mathscr{R}^{\mathbf{N}}$ can be regarded as two-dimensional sequences $r_{n,j}$, n, $j \in \mathbf{N}$, of zeros and ones. Let

u be $\frac{1}{2}$, $\frac{1}{2}$ product measure on all these zeros and ones. I will write P instead of u, connoting probability of.

Now fix n_0, j_0 and suppose we simply alter r_{n_0,j_0} to $\hat{r}_{n_0,j_0} = 1 - r_{n_0,j_0}$ keeping all other $r_{n,j}$ fixed. The output $\{b_i\}$ in $\{0,1\}^N$ will alter to $\{\hat{b}_i\}$, $i \in \mathbf{N}$.

Now change all $r_{n_0,j}$ to zero and thereby alter all $\{b_i\}$ to $\hat{\hat{b}}_i$. By property 1 there exists $g(n_0)$ such that

$$P(\text{ for all } i > g(n_0), \quad b_i = \hat{\hat{b}}_i) > 1 - \frac{1}{2 \cdot (100)^{n_0}}.$$

By symmetry

$$P(\text{for all } i > g(n_0), \quad \hat{b}_i = \hat{\hat{b}}_i) > 1 - \frac{1}{2 \cdot (100)^{n_0}}.$$

Therefore

$$(3) \qquad P(\text{for all } i > g(n_0), \quad b_i = \hat{b}_i) > 1 - \left(\frac{1}{100}\right)^{n_0}.$$

Note that $g(n_0)$ depends only on n_0 and not on j_0.

By property 2, for every n_0 there exists n_1 such that $b_1, b_2, \ldots, b_{g(n_0)}$ will all be practically determined by $\{r_{i,j} : i \leq n_1, \text{ and } j \leq n_1\}$. Thus fixing n_0, choosing n_1 sufficiently large, choosing $j_0 > n_1$, and as before, switching r_{n_0,j_0} to $r_{n_0,j_0} = 1 - r_{n_0,j_0}$ we get

(4) $P(\text{for } i \in \{1, 2, \ldots, g(n_0)\}, \quad b_i = \hat{b}_i) > 1 - (\frac{1}{100})^{n_0}$.

3 and 4 yield

(5) P (for every n_0, there exists j_0, such that when r_{n_0,j_0} is altered to $1 - r_{n_0,j_0}$ the output is not altered) $> 1 - 2(\frac{1}{100})^{n_0}$.

Summing up over n_0 we get that there is positive probability after altering all r_{n_0}, that the output will remain unchanged. This contradicts condition 1.

2. Good maps exist

A map from one collection of sequences to another collection of sequences is called "good" if it obeys the conditions of "good" in the Introduction. Clearly, the property "good" is closed under composition.

Let $\mathbf{R}^N$ be sequences of reals.

Let $\aleph_1^N$ be sequences of ordinals less than $\aleph_1$.

Let $\widehat{\aleph_1^N}$ be sequences of ordinals less than $\aleph_1$ without repetition.

Let L be sequences of finite sequences of countable ordinals $\{L_i\}_{i=1}^\infty$ such that all but finitely many of the countable ordinals which occur in $\bigcup_{i=1}^\infty L_i$ occur in infinitely many of the L_i.

Let $\mathbf{N}^N$ be sequences of non-negative integers.

Let $\{0,1\}^N$ be sequences of zeros and ones.

We will show that there is a good map from $\mathbf{R}^N$ to $\{0,1\}^N$ by showing that there is a good map from $\mathbf{R}^N$ to $\aleph_1^N$, from $\aleph_1^N$ to $\widehat{\aleph_1^N}$, from $\widehat{\aleph_1^N}$ to L, from L to $\mathbf{N}^N$, and from $\mathbf{N}^N$ to $\{0,1\}^N$:

(a) From $\mathbf{R}^N$ to $\aleph_1^N$. Using the continuum hypothesis $\mathbf{R}^N$ and $\aleph_1^N$ can be regarded as identical. This is the only use of the continuum hypothesis in this paper.

(b) From $\aleph_1^N$ to $\widehat{\aleph_1^N}$. Let Φ be a bijection from $\aleph_1 \times \mathbf{N}$ to $\aleph_1$; the map $\varnothing: \aleph_n^N \to \widehat{\aleph_1^N}$ defined by $(a_1, a_2, \dots,) \mapsto (\Phi(a_1, 1), \Phi(a_2, 2)\dots)$ is good.

(c) From $\widehat{\aleph_1^N}$ to L.

The following proof is due to the referee. It is simpler than my original proof.

For each ordinal $\alpha < \aleph_1$, let D_n^α be a nondecreasing sequence of finite sets whose union is α. Given any sequence $\{\alpha_n\}_{n=1}^\infty$, then $\varnothing(\{\alpha_n\}_{n=1}^\infty) = \{F_n\}_{n=1}^\infty$ where F_n is a list in increasing order of the smallest set of ordinals F, such that $\{\alpha_m: m \le n, \alpha_m \le \alpha_n\} \subseteq F$ and for any $B \in F$, $D_n^B \subseteq F$. We now prove $\varnothing$ to be a good map. This means that if $\varnothing(\{\alpha_n\}_{n=1}^\infty) = \{F_n\}_{n=1}^\infty$ and $\varnothing(\{\beta_n\}_{n=1}^\infty) = \{G_n\}_{n=1}^\infty$ then

(i) If $\alpha_n \ne \beta_n$ for infinitely many n, then $F_n \ne G_n$ for infinitely many n.

(ii) If $\alpha_n \ne \beta_n$ for only finitely many n, then $F_n \ne G_n$ for only finitely many n.

(iii) F_n depends only on $\alpha_1, \alpha_2, \dots, \alpha_n$.

Proof of (i). For a fixed n, suppose $\alpha_n \ne \beta_n$. Suppose W.L.O.G. $\alpha_n > \beta_n$. Then $\alpha_n \in F_n$ but $\alpha_n \notin G_n$ so $F_n \ne G_n$.

Proof of (ii). There exists N_1 such that $\alpha_n = \beta_n$ for all $n \ge N_1$. Because there is no repetition on $\{\alpha_n\}$ or $\{\beta_n\}$, there exists N_2 so that for all $n \ge N_2$, $\alpha_n < \overline{\lim}\,\alpha_i = \overline{\lim}\,\beta_i$ and $\beta_n < \overline{\lim}\,\alpha_i = \overline{\lim}\,\beta_i$.

For any $\gamma < \overline{\lim}\,\alpha_i$, let $f(\gamma)$ be the least member of the set

$$\{\alpha_i: i > \max(N_1, N_2) \text{ and } \alpha_i > \gamma\}.$$

For each $\gamma < \overline{\lim}\,\alpha_i$ let $g(\gamma) = n$, where n is chosen so that $\{\alpha_i: i \le \max(N_1, N_2) \wedge \alpha_i < \gamma\} \cup \{\beta_i: i \le \max(N_1, N_2) \wedge \beta_i < \gamma\} \subset D_n^\gamma$. I claim that $F_n = G_n$ for all $n >$

$$\max(\{m: \exists i, \alpha_m = f(\alpha_i) \wedge i \le \max(N_1, N_2) \wedge \alpha_i < \overline{\lim}\,\alpha_j\}$$

$$\cup \{m: \exists i, \beta_m = f(\beta_i) \wedge i \le \max(N_1, N_2) \wedge \beta_i < \overline{\lim}\,\beta_j\}$$

$$\cup \{g(f(\alpha_i)): i < \max(N_1, N_2), \alpha_i < \overline{\lim}\,\alpha_j\}$$

$$\cup \{g(f(\beta_i)): i < \max(N_1, N_2), \beta_i < \overline{\lim}\,\beta_j\} \quad \text{note } \overline{\lim}\,\alpha_i = \overline{\lim}\,\beta_j).$$

Fix n as above. Let $S = \{a_m: m \le n \text{ and } a_m \le a_n\}$. We are given that $S \subset F_n$. The reader can verify that to prove $F_n \subset G_n$ it suffices to prove that

$S \subset G_n$. We prove $G_n \subset F_n$ similarly. It is trivial by the definition of N_1 that $S \cap \{\alpha_m : m \geq N_1\} \subset G_n$. We need only prove that $S \cap \{\alpha_m : m \leq N_1\} \subset G_n$. Since $S \cap \{a_m : m < N_1\}$ is finite, choose the largest element α_j, $f(\alpha_j) \in S \cap \{\alpha_m : m \geq N_1\}$ so $f(\alpha_j) \in G_n$. Therefore $D_n^{f(\alpha_j)} \subset G_n$. Since $n > g(f(\alpha_j))$ it follows that $S \cap \{\alpha_m : m < N_1\} \subset D_n^{f(\alpha_j)} \subset G_n$. $\square$

Proof of (iii). Obvious. $\square$

From L to $\mathbf{N}^{\mathbf{N}}$.

Let $\{L_j\}_{i=1}^{\infty} \in L$. We wish to construct a function which takes $\{L_i\}_{i=1}^{\infty}$ to a sequence $\{n_i\}_{i=1}^{\infty}$ in $\mathbf{N}^{\mathbf{N}}$. To see how this function is to be constructed, suppose $L_{17} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. Here is some information I am going to require:

(i) I will require knowledge of the number 5 (the number of terms in L_{17}).
(ii) Since this is L_{17}, I will require knowledge of the first 17 digits of α_1, of α_2, of α_3, of α_4, and of α_5.
(iii) I will require knowledge of whether each term of L_{17} occurred before, when it occurred last, and in what position.

Remark. Maybe α_3 never occurred before. I require that I know that. Maybe the last time α_4 occurred was in L_{10}, and there it occurred as the twelfth term. I require that I know that.

Above is a collection of information that I require that I must know. Clearly, all this information can be encoded in an integer. *We define n_{17} to be the integer which encodes the information I want to know.* In this manner we define $\{n_i\}_{i=1}^{\infty}$. We now must establish

(i) If we only alter finitely many of the L_i, then we only alter finitely many of the n_i.
(ii) If we alter infinitely many of the L_i then we alter infinitely of the n_i.

Proof of (i). Suppose we only alter L_i for some of the i where $i < 10$. For each i, let $A_i = L_i$ if L_i is not altered, and let A_i be the altered L_i if L_i is altered. For any $\alpha \in \bigcup_{i=1}^{10} L_i \cup A_i$ define $F(\alpha)$ to be the least $i > 10$ where $\alpha \in L_i$ ($F(\alpha) = 10$ if no such i exists). It is easy to see that no value of n_i is altered for any $i > \sup\{F(\alpha) | \alpha \in \bigcup_{i=1}^{10} L_i\}$.

Proof of (ii). Our procedure will be to merely show that if we alter L_{17} then one of the n_i will be altered. The reader will then be able to easily see (ii) for himself.

Suppose $\{L_i\}_{i=1}^{\infty}$ and $\{\widehat{L}_i\}_{i=1}^{\infty} \in L$ are not identical. In particular suppose $L_{17} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $\widehat{L}_{17} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4, \hat{\alpha}_5)$.

Let us presume that α_4 and $\hat{\alpha}_4$ differ on the 1000th digit. We do not specify whether or not there are any other differences between $\{L_i\}_{i=1}^{\infty}$ and $\{\widehat{L}_i\}_{i=1}^{\infty}$ other than that one digit of α_4. We claim that the sequences in $\mathbf{N}^{\mathbf{N}}$ that they code to must be different. We use proof by contradiction. We will assume that they both code to the same $\{n_i\}_{i=1}^{\infty} \in \mathbf{N}^{\mathbf{N}}$.

The proof does not necessarily work (i.e. no contradiction is obtained) if α_4 only occurs finitely many times in $\{L_i\}_{i=1}^{\infty}$; however since there are only finitely many such values α in $\bigcup_{i=1}^{\infty} L_i$ and since we will eventually want to change infinitely many of the L_i, this possibility need not be considered.

In other words, we can assume α_4 occurs infinitely many times in $\{L_i\}_{i=1}^{\infty}$. Let us assume that the next time α_4 occurs, after L_{17}, is as the 13th member of L_{32}, which will be denoted by α_{13}^{32}. $\alpha_{13}^{32} = \alpha_4$ and the fact that $\alpha_{13}^{32} = \alpha_4$ is encoded by n_{32}. Since n_{32} is also the 32nd term coded to by $\{\widehat{L}_i\}_{i=1}^{\infty}$ it follows that the 13th term of $\widehat{L}_{32}$ (which we denote by $\hat{\alpha}_{13}^{32}$) equals $\hat{\alpha}_4$. $\hat{\alpha}_{13}^{32} = \hat{\alpha}_4$. Now suppose that the next time α_4 occurs, after L_{32} is as the eighth term of L_{52} which we denote as α_8^{52}. Now $\alpha_8^{52} = \alpha_4 = \alpha_{13}^{32}$ so by reasoning we just used, the eighth term of $\widehat{L}_{52}$, $\hat{\alpha}_8^{52}$, exists and equals $\hat{\alpha}_{13}^{32}$. Therefore $\hat{\alpha}_8^{52} = \hat{\alpha}_{13}^{32} = \hat{\alpha}_4$. Continuing this reasoning over and over again we eventually get a number over 1000 (say 1035) and some arbitrary number (say 15) such that if we let α_{15}^{1035} and $\hat{\alpha}_{15}^{1035}$ be the 15th terms of L_{1035} and $\widehat{L}_{1035}$, respectively, then we can prove $\alpha_{15}^{1035} = \alpha_4$ and $\hat{\alpha}_{15}^{1035} = \hat{\alpha}_4$. Since both $\{L_i\}_{i=1}^{\infty}$ and $\{\widehat{L}_i\}_{i=1}^{\infty}$ code to the same n_{1035}, it follows that α_{15}^{1035} and $\hat{\alpha}_{15}^{1035}$ have the same first 1035 digits. Thus α_4 and $\hat{\alpha}_4$ have the same 1035 digits. This is false. α_4 and $\hat{\alpha}_4$ differ on their 1000th digit.

(e) From $\mathbf{N}^{\mathbf{N}}$ to $\{0,1\}^{\mathbf{N}}$.

Let $\{n_i\}_{i=1}^{\infty} \in \mathbf{N}^{\mathbf{N}}$. For all pair i, $j \in \mathbf{N}$, let

$$a_{i,j} = \begin{cases} 1 \text{ if } \quad j = n_i \\ 0 \text{ else.} \end{cases}$$

We define $F: \mathbf{N}^{\mathbf{N}} \to \{0,1\}^{\mathbf{N}}$ by

$$F(\{n_i\}_{i=1}^{\infty}) = (a_{0,0}, a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}, a_{3,0}, \dots).$$

The reader can easily verify himself that t is good. $\square$

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