A REMARK ON FINITELY PRESENTED INFINITE DIMENSIONAL ALGEBRAS

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(Communicated by Donald S. Passman)

ABSTRACT. By estimating dimensions of representation varieties, we show that certain finitely presented algebras are infinite dimensional.

The object of this note is to use a tiny sliver of the geometry of representations of a finitely generated algebra to prove the following

**Theorem.** Let $A$ be an augmented algebra (over a field $k$) with augmentation ideal $\mathcal{A}$, given by the finite presentation

$$A = \langle x_1, \ldots, x_m; w_1^q, \ldots, w_n^q \rangle \quad (w_i \in \mathcal{A}, q \geq 2).$$

If

$$n \leq (m - 1)q$$

then $A$ is infinite dimensional.

Our theorem should be viewed in the light of the following theorem of J. Levitzki [3]: if every element of the augmentation ideal of a finitely generated augmented algebra $A$ is nilpotent of bounded degree, then $A$ is finite dimensional. It also can be viewed as a counterpart to a similar theorem about finitely presented groups that we proved a couple of years ago [1].

Before we embark on the proof of the theorem we observe first that we lose nothing if we assume that $k$ is algebraically closed. The basic idea is to make use of the parametrisation of the set $X(A, q)$ of all the equivalence classes of semisimple representations of $A$ in $M(q, k)$, the $k$-algebra of all $q \times q$ matrices over $k$, given by Procesi [4]. We recall the details of this parametrisation in a form suitable for the purposes we have in mind. To this end, suppose that $F$ is the free associative $k$-algebra on $x_1, \ldots, x_m$, and that $U = M(q, k)^m$. We associate to each representation $\rho$ of $F$ the point

$$u = (\rho(x_1), \ldots, \rho(x_m)) \in U.$$
Each element \( w = w(x_1, \ldots, x_m) \in F \) defines \( q \) polynomial functions \( f_w^i \) \((i = 0, \ldots, q - 1)\) on \( U \), as follows:

\[
f_w^i(u) = \text{the coefficient of the degree } i \text{ term of the characteristic polynomial of } \rho(w), \text{ where } \rho \text{ is the representation of } F \text{ in } M(q, k) \text{ corresponding to } u.\]

Then it turns out that the \( k \)-subalgebra \( B \) of \( F \) generated by these polynomial functions \( f_w^i \), where \( w \) ranges over \( F \) and \( i = 0, \ldots, q - 1 \), is a finitely generated subalgebra of the \( k \)-algebra \( P \) of all polynomial functions on \( U \). Notice that \( P \) is the \( k \)-algebra of polynomials in \( mq^2 \) variables. Now let \( X(F, q) \) be the affine algebraic set corresponding to this algebra \( B \) and let \( p \) be the canonical map from \( U \) into \( X(F, q) \). Then Procesi [4] proves that the following hold:

1. \( p \) is onto \( X(F, q) \);
2. if \( S \) is the subset of \( U \) consisting of semi-simple representations of \( F \), then \( p \) maps \( S \) onto \( X(F, q) \);
3. if \( \rho \in S \) is irreducible, then \( p^{-1}(\rho) \) is the set of all representations of \( F \) in \( M(m, q) \) equivalent to \( \rho \).

It follows, in particular, that if \( \rho \) is an irreducible representation of \( F \), then \( p^{-1}(\rho) \) is of dimension \( q^2 - 1 \). Now \( X(F, 2) \) is an affine variety, i.e. it is irreducible. So it follows (see e.g. Humphreys [2, page 30]) that

\[
\dim(X(F, q)) \geq \dim(R(F, q)) - \dim(p^{-1}(\rho)) = mq^2 - (q^2 - 1) = (m - 1)q^2 + 1.
\]

Now if \( M \) is a matrix of degree \( q \) over \( k \), then \( M^q = 0 \) if and only if its characteristic polynomial is \( t^q \). This means that the coefficients of all of the powers of \( t \) except for \( t^q \) in the characteristic polynomial of \( M \) are zero. We need to apply this remark to the defining relations of \( A \). Observe then that \( \rho(u_j^q) = 0 \) for every representation \( \rho \) of \( A \) in \( M(q, k) \) if and only if the functions \( f_{u_j}^i = 0 \), for \( i = 0, \ldots, q - 1 \). The existence of at least one such representation is guaranteed by the hypothesis since \( A \) is an augmented algebra. Consider then these functions \( f_{u_j}^i = g_j^i \) \((i = 0, \ldots, q - 1, j = 1, \ldots, n)\). Every such function \( g_j^i \) lies in the algebra \( B \). Consequently they can be viewed as polynomial functions on \( X(F, q) \) with values in \( k \). Let \( h_j^i = p \circ f_j^i \). Consider

\[
V = \bigcap_{j, i}(h_j^i)^{-1}(0).
\]

This is therefore an affine algebraic set, since the \( h_j^i \) are polynomial functions. And

\[
p(V) = \bigcap_{j, i}(g_j^i)^{-1}(0).
\]
is therefore also an affine algebraic set in $X(F,d)$ whose dimension is then bounded as follows:

$$\dim p(V) \geq (m - 1)q^2 + 1 - nq.$$ 

Now every point in $p(V)$ corresponds to an equivalence class of semi-simple representations of $F$ such that for each representation $p \in V$ the characteristic polynomial of every $p(w_j)$ is $t^q$. So by the remark above, $p$ factors through $A$, yielding a semi-simple representation of $A$. This means that $p(V)$ parametrises a set of inequivalent semi-simple representations of $A$. Since a finite dimensional algebra has only finitely many inequivalent representations overall, if $\dim p(V) > 0$ then $A$ is infinite dimensional. This is precisely what the inequality given in the theorem ensures.

REFERENCES


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