FIXED POINT THEOREMS IN PRODUCT SPACES

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ABSTRACT. If $K_1$ and $K_2$ are nonempty closed weakly compact subsets of Banach spaces and they have the generic fixed point property for nonexpansive mappings, then in the maximum norm $K_1 \times K_2$ has fixed point property for nonexpansive mappings.

In this paper we study fixed points of nonexpansive mappings in product spaces.

Let $(K, \zeta_1)$ and $(L, \zeta_2)$ be metric spaces. A mapping $T: K \to L$ is nonexpansive if

$$\zeta_2(Tx, Ty) \leq \zeta_1(x, y)$$

for all $x, y \in K$. Let $\chi$ denote the Hausdorff (or ball) measure of noncompactness in $K$ and $L$. A continuous mapping $T: K \to L$ is a $k$-set contraction $k \in [0, 1))$, if for any $K_0 \subset K$ we have

$$\chi(TK_0) \leq k\chi(K_0).$$

$(K, \zeta_1)$ has a fixed point property for nonexpansive (continuous) mappings if every nonexpansive (continuous) self-mapping $T: K \to K$ must have a fixed point.

Let $K$ be a nonempty, convex, weakly compact subset of a Banach space $X$. $K$ has the generic fixed point property for nonexpansive mappings if for every nonexpansive self-mapping $T: K \to K$ and every nonempty, convex, closed subset $K_0 \subset K$ with $TK_0 \subset K_0$, we have $K_0 \cap \text{Fix}(T) \neq \emptyset$, where $\text{Fix}(T) = \{x \in K: Tx = x\}$.

Suppose that $(X, \| \cdot \|_1)$ is a Banach space and $(K_2, \zeta_2)$ is a metric space. Let $X \times K_2$ denote a product space of $X$ and $K_2$ with a maximum metric

$$\zeta_{\infty}((x, u), (y, v)) = \max(\|x - y\|_1, \zeta_2(u, v))$$

for $x, y \in X$ and $u, v \in K_2$. It was shown in [7] (see also [2, 5, 6, 8, 9, 10]) that if $X$ has $KK$-norm, $\emptyset \neq K_1 \subset X$ is weakly compact and convex, and $K_1$ and $K_2$ have the fixed point property for nonexpansive mappings, then every

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nonexpansive (with respect to the metric \( \zeta \)) \( T : K_1 \times K_2 \to K_1 \times K_2 \) has a fixed point. Here we generalize this result. We begin with

**Theorem 1.** Let \( (X, \| \|_1) \) be a Banach space and let \( (K_2, \zeta) \) be a metric space. Suppose that \( \emptyset \neq K_1 \subset X \) is weakly compact, convex and has the generic fixed point property for nonexpansive mappings. If \( F : K_1 \times K_2 \to K_1 \) is a nonexpansive mapping, then there exists a nonexpansive mapping \( R : K_1 \times K_2 \to K_1 \) such that

\[
F(R(x, u), u) = R(x, u)
\]

for \( (x, u) \in K_1 \times K_2 \) and \( R(x, u) = x \) when \( F(x, u) = x \).

**Proof.** We construct a mapping \( \tilde{F} : K_1 \times K_2 \to K_1 \times K_2 \) in the following way

\[
\tilde{F}(x, u) = (F(x, u), u) = (F, P_2)(x, u)
\]

for \( (x, u) \in K_1 \times K_2 \), where \( P_2 \) is the second coordinate projection. It is trivial to check that \( \tilde{F} \) is nonexpansive and \( A = \text{Fix}(\tilde{F}) \neq \emptyset \). Now we define the set

\[
N(A) = \{ (G, P_2) \in K_1 \times K_2 : G : K_1 \times K_2 \to K_1 \text{ is nonexpansive and } A \subset \text{Fix}(G) \}.
\]

Note that

\[
N(A) \subset \prod_{(x, u) \in K_1 \times K_2} K_1 \times \{ u \} \sim \prod_{(x, u) \in K_1 \times K_2} K_1.
\]

In \( K_1 \) we have the weak topology, so by Tychonoff's Theorem \( \prod_{(x, u) \in K_1 \times K_2} K_1 \) is compact in the product topology. This topology can be transposed on the set \( \prod_{(x, u) \in K_1 \times K_2} K_1 \times \{ u \} \) and \( N(A) \) is closed in this topology. It allows us to apply Bruck's method ([1]) step-by-step to obtain a nonexpansive retraction \( r : K_1 \times K_2 \to A \) which additionally belongs to \( N(A) \), i.e. \( r = (R, P_2) \). \( R \) satisfies the desired conditions.

Let \( X, K_1, K_2 \) be as in Theorem 1. Suppose that \( T = (T_1, T_2) : K_1 \times K_2 \to K_1 \times K_2 \) is a mapping such that \( T_1 : K_1 \times K_2 \to K_1 \) is nonexpansive. Fix \( x_0 \in K_1 \). The mapping \( \overline{T_2} : K_2 \to K_2 \) is defined by

\[
\overline{T_2}(u) = T_2(R(x_0, u), u), \quad u \in K_2,
\]

where \( R \) satisfies Theorem 1. It is clear that \( \overline{T_2}(u) = u \) implies that \( T(R(x_0, u), u) = (R(x_0, u), u) \). Let \( K_2 \) have the fixed point property with respect to a subclass \( S \) of \( K_2^{K_1} \). If \( \overline{T_2} \in S \), then \( \text{Fix}(T) \neq \emptyset \).

**Theorem 2.** Let \( X, K_1, K_2 \) be as in Theorem 1 and let \( T_1 : K_1 \times K_2 \to K_1 \) be nonexpansive.

(i) If \( (K_2, \zeta) \) has a fixed point property for nonexpansive mappings and \( T_2 : K_1 \times K_2 \to K_2 \) is nonexpansive, then \( T = (T_1, T_2) : K_1 \times K_2 \to K_1 \times K_2 \) has a fixed point.

(ii) If \( (K_2, \zeta) \) has a fixed point property for continuous mappings and \( T_2 : K_1 \times K_2 \to K_2 \) is continuous, then \( T = (T_1, T_2) : K_1 \times K_2 \to K_1 \times K_2 \) has a fixed point.
(iii) If $\emptyset \neq K_2$ is a convex, closed subset of a Banach space $X$ and $T_2: K_1 \times K_2 \to K_2$ is a $k$-set contraction with respect to Hausdorff measure of noncompactness, then $T = (T_1, T_2): K_1 \times K_2 \to K_1 \times K_2$ has a fixed point.

Proof. It is sufficient to observe that in the first case $S$ is the set of all nonexpansive self-mappings in $K_2$, in the second one $S$ is the set of all continuous self-mappings in $K_2$, and in the third one $S$ is the set of all $k$-set contractions in $K_2$.

Remark 1. If $K_1, \ldots, K_n$ are nonempty, convex, weakly compact subsets of Banach spaces and have the generic fixed point property for nonexpansive mappings, then $\prod_{i=1}^{n} K_i$ has a fixed point property for nonexpansive mappings (with respect to the maximum norm).

Remark 2. For discussion of spaces which have the fixed point property with respect to nonexpansive mappings, we refer to Kirk and ([3, 4]) and Reich ([11, 12]).

Remark 3. If $X_1$ is the conjugate Banach space, then in the above theorems the weak topology can be replaced by the weak-\* topology.

References