A PROBLEM IN ELECTRICAL PROSPECTION
AND A n-DIMENSIONAL BORG-LEVINSON THEOREM

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Abstract. We show that the Dirichlet to Neumann map for \(-\Delta u + vu = 0\), determines the potential \(v(x)\), for \(v(x)\) satisfying the condition of C. Fefferman and D. Phong.

We shall consider here a bounded domain \(\Omega \subset \mathbb{R}^n\), \(n \geq 3\) with smooth boundary. Consider then the equation in \(\Omega\) given by \(-\Delta u + vu = 0\). Define the Dirichlet to Neumann map \(\Lambda_v\) on \(\partial \Omega\) given by

\[
\Lambda_v(f) = \frac{\partial u}{\partial v},
\]

and \(u\) solves the Dirichlet problem \(-\Delta u + vu = 0\) in \(\Omega\) and \(u|_{\partial \Omega} = f\).

We recall, ([F], [CW]) the definition of the C. Fefferman, D. Phong class. We say \(v \in F_p\) if for all cubes \(Q \subset \mathbb{R}^n\),

\[
\|v\|_{F_p} = \sup_Q |Q|^{2/n} \left[ \frac{1}{|Q|} \int_Q |v|^p \right]^{1/p} < \infty.
\]

We remark that \(L^{n/2}(\mathbb{R}^n) \subset F_p\) for \(p \leq n/2\), and likewise \(L^{n/2, \infty} \subset F_p\), \(p < n/2\). The containments are strict as \(v = f(x/|x|)|x|^{-2}\), \(f \in L^p(S^{n-1})\), \(p > (n-1)/2\) is not in \(L^{n/2, \infty}\) but \(v \in F_p\), \(p > (n-1)/2\). The main result proved here is as follows.

Theorem. Suppose \(\|v_i\|_{F_p} \leq \varepsilon(n),\ p > (n-1)/2,\ i = 1,2\). Assume that \(\Lambda_{v_1} = \Lambda_{v_2}\), then \(v_1 = v_2\) in \(\Omega\).

Remark. In the theorem above it is enough to assume that \(v_i\) are supported in \(\Omega\).

The one-dimensional result is due to [B], [L]. If \(v_i \in L^\infty\) the result is due to [NSU] and [HN]. The two-dimensional result is in [SU,1] and the \(C^\infty\) case in

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Applications to conductivity measurements are in [C], [KV], [SU$_1$] and [SU$_2$]. [KV] also treats the case where $v_i$ are analytic.

The smallness assumption on the $F_p$ norm is not needed if $v_i \in L^p(\Omega)$, $p > n/2$. We wish to thank D. Jerison and C. Kenig for pointing this out to us and also include their proof of this observation after the end of the proof in the main theorem above.

We recall the following inequality from [CS].

**Theorem 1.** Let $f \in C_0^\infty(\mathbb{R}^n)$, $v > 0$ and $v \in F_p$, $p > (n-1)/2$. For $z \in C^n, \gamma \in C$, define $Q(D) = \Delta + z \cdot \nabla + \gamma$. Then

$$\int_{\mathbb{R}^n} |f|^2 v \leq c \int_{\mathbb{R}^n} |Q(D)f|^2 v^{-1}$$

where $c$ is independent of $f$, $z$, $\gamma$.

We use the theorem stated above to prove the lemma that follows.

**Lemma 1.** Let $v_i \in F_p \cap L^1$, $p > (n-1)/2$, and $\|v_i\|_{F_p} \leq \epsilon$, $i = 1, 2$. Let $z \in C^n$ with $z \cdot z = 0$. Let $V(x) = |v_1| + |v_2| + \delta(1 + |x|^2)^{-n}$, $\delta > 0$ and small. Let $L^2_V = \{f : \int_{\mathbb{R}^n} |f|^2 V < \infty\}$. Then,

(a) there is a unique solution to $-\Delta + v_i$ of the form,

$$u_i(x) = e^{z \cdot x} m_{z,i}(x), i = 1, 2$$

with $m_{z,i}$ in the space $L^2_V$.

(b) $\int_{\mathbb{R}^n} |m_{z,i}|^2 V \leq c$, uniformly in $z$, $i = 1, 2$.

(c) $m_{z,i}(x) \to 1$, $i = 1, 2$ weakly in $L^2_V$ as $|z_k| \to \infty$, for some sequence $z_k$.

**Proof.** Substituting $u = e^{z \cdot x} m_{z,i}(x)$ into $-\Delta u + v_i u = 0$, we note that $m_{z,i}(x)$ satisfies the equation

$$-\Delta m_{z,i} + (z \cdot \nabla)m_{z,i} + v_i m_{z,i} = 0.$$ 

Therefore, $m_{z,i}$ satisfies the integral equation,

$$m_{z,i} = 1 + G_z(v_i m_{z,i}), \quad i = 1, 2$$

where $G_z$ denotes the Green function for $-\Delta + z \cdot \nabla$. Define $T_i f(x) = 1 + G_z(v_i f)(x)$. It will be enough for us to show $T_i$ has a fixed point on the Banach space $L^2_V$, thus showing (a). In fact we show $T_i$ is a contraction on $L^2_V$ and thus the uniqueness assertion (a) of Lemma 1 also follows. Since $v_1, v_2 \in L^1$, $V \in L^1$. Since $v_1, v_2$, $\delta(1 + |x|^2)^{-n} \in F_p$, $p > (n-1)/2$, we have $\|V\|_{F_p} \leq \epsilon$, for $p > (n-1)/2$. Moreover $V > 0$. Thus, Theorem 1 is applicable with $v = V$. Moreover $|v_i| \leq V$, $i = 1, 2$. 

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Thus,
\[ \int_{\mathbb{R}^n} |T_i f|^2 V \leq \int_{\mathbb{R}^n} V + \int_{\mathbb{R}^n} |G_z(v_i f)|^2 V \]
\[ \leq c + \varepsilon \int_{\mathbb{R}^n} |v_i f|^2 V^{-1} \leq c + \varepsilon \int_{\mathbb{R}^n} |f|^2 V^2 V^{-1} \]
\[ \leq \int_{\mathbb{R}^n} |f|^2 V < \infty. \]

Similarly,
\[ \int_{\mathbb{R}^n} |T_i (f - g)|^2 V \leq \int_{\mathbb{R}^n} |G_z(v_i (f - g))|^2 V \leq \varepsilon \int_{\mathbb{R}^n} |f - g|^2 V^{-1} \]
\[ \leq \varepsilon \int_{\mathbb{R}^n} |f - g|^2 V. \]

Thus \( T_i \) is a contraction and the existence and uniqueness of \( m_{z,i} \) is assured.

We now show (b). By (1),
\[ \int_{\mathbb{R}^n} |m_{z,i}|^2 V \leq c \int_{\mathbb{R}^n} V + c \int_{\mathbb{R}^n} |G_z(v_i m_{z,i})|^2 V \]
\[ \leq c_1 \int_{\mathbb{R}^n} V + \varepsilon c_2 \int_{\mathbb{R}^n} |m_{z,i}|^2 V \]
where \( c_1, c_2 \) do not depend on \( z \). Thus for small \( \varepsilon \), and since from (a), \( \int_{\mathbb{R}^n} |m_{z,i}|^2 V < \infty \), we see,
\[ \int_{\mathbb{R}^n} |m_{z,i}|^2 V \leq c, \quad \text{uniformly in } z. \]

We now prove (c). Note the multiplier for \( G_z \) given by \( (|\xi|^2 + iz \cdot \xi)^{-1} \to 0 \) as \( |z| \to \infty \). Next we note by (b),
\[ \int_{\mathbb{R}^n} |G_z(v_i m_{z,i})|^2 V \leq \varepsilon \int_{\mathbb{R}^n} |m_{z,i}|^2 V \leq c. \]

Thus there is a sequence \( z_k, |z_k| \to \infty \), so that \( G_{z_k}(v_i m_{z_k,i}) \to 0 \) weakly in \( L^2_V \). Since (1) holds, it follows that \( m_{z_k,i} \to 1 \) as \( |z_k| \to \infty \) in \( L^2_V \). The lemma is now proved.

**Lemma 2.** Extend \( v_1 \) and \( v_2 \) to be zero outside \( \Omega \). Let \( u_i \) be the unique solutions of Lemma 1 to \( -\Delta + v_i, i = 1, 2 \). If \( \Lambda_{v_1} = \Lambda_{v_2} \), then \( u_1 = u_2 \) in \( \mathbb{R}^n \setminus \Omega \).

**Proof.** Recall, [F], [CW], that if \( \|v_i\|_{F_p} \leq \varepsilon, p > 1 \), then for \( f \in C_0^\infty(\mathbb{R}^n) \),
\[ \int_{\mathbb{R}^n} |f|^2 |v_i| \leq \varepsilon \int_{\mathbb{R}^n} |\nabla f|^2. \]

Thus the bilinear form,
\[ \int_{\Omega} \nabla u \cdot \nabla \varphi + \int_{\Omega} u \varphi v_2 \]
is coercive and continuous for $u$, $\varphi \in H^1_0(\Omega)$. As,
\[
\int_{\Omega} |m_{z,1}|^2 |v_2| \leq \int_{\mathbb{R}^*} |m_{z,1}|^2 |v_2| \leq \int_{\mathbb{R}^*} |m_{z,1}|^2 V < \infty,
\]
and $u_1 = e^{z \cdot x} m_{z,1}$, we get $\int_{\Omega} |u_1|^2 |v_2| < \infty$. Thus, the Dirichlet problem,
\[
-\Delta u + uv_2 = 0 \quad \text{in } \Omega
\]
\[
u = u_1 \quad \text{on } \partial \Omega
\]
has a unique solution $u$, such that $\int_{\Omega} |u|^2 |v_2| < \infty$.

Since $u - u_1$ has compact support, by (2), as $V \in F_p$, $p > 1$,
\[
\int_{\Omega} |u - u_1|^2 V \leq \int_{\Omega} |\nabla (u - u_1)|^2 < \infty.
\]
On $\Omega$, $|u_1| \leq c|m_{z,1}|$, thus, $\int_{\Omega} |u|^2 V < \infty$ by Lemma 1.

Define,
\[
\Phi = \begin{cases} u & \text{in } \Omega \\ u_1 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}
\]
Since $\Lambda_{\nu_1}(u_1) = \partial u / \partial \nu = \Lambda_{\nu_1}(u_1) = \partial u_1 / \partial \nu$, $\Phi$ is a solution to $\mathbb{R}^n$ to $-\Delta + v_2$.
Writing $\Phi = e^{z \cdot x}[e^{-z \cdot x} \Phi] = e^{z \cdot x} M_z(x)$, we see that $M_z(x) = m_{z,1}(x)$ in $\mathbb{R}^n \setminus \Omega$, and since $\int_{\Omega} |u|^2 V < \infty$, it follows that $\int_{\mathbb{R}^*} |M_z|^2 V < \infty$. By the uniqueness assertion of Lemma 1, $M_z = m_{z,2}$, and thus $\Phi = u_2$, in particular $u_1 = u_2$ in $\mathbb{R}^n \setminus \Omega$.

We are now in a position to prove our main theorem.

**Proof.** Fix $l \in \mathbb{Z}^n$. Choose $k, e \in \mathbb{R}^n$, so that $|k| = |l-e|$, $k \cdot e = k \cdot l = e \cdot l = 0$. This choice forces $|k| = |l+e|$. Let $z = \frac{1}{2}(-k+i(l-e))$, $\bar{z} = \frac{1}{2}(k+i(l+e))$. Note $z \cdot z = \bar{z} \cdot \bar{z} = 0$. We shall use Green's theorem in the form,
\[
\int_{\Omega} (w \Delta f - f \Delta w) = \int_{\Omega} \left[ w \frac{\partial f}{\partial \nu} - f \frac{\partial w}{\partial \nu} \right] d\sigma
\]
with the choice $w = e^{z \cdot x}$, $f = u_1 = e^{z \cdot x} m_{z,1}$. Let $D_\rho$ be a collar neighborhood of $\partial \Omega$ with thickness $\rho$. Let $\partial D_\rho \cap \Omega = \partial D_{\rho,1}$ and $\partial D_\rho \cap (\mathbb{R}^n \setminus \Omega) = \partial D_{\rho,2}$. Note
\[
\int_{\Omega} e^{z \cdot x} v_j u_i = -\int_{\partial \Omega} e^{z \cdot x} \Delta u_j = \int_{\partial \Omega} e^{z \cdot x} \left[ (\bar{z} \cdot \nu) u_i - \frac{\partial u_i}{\partial \nu} \right] d\sigma.
\]
We now show that,
\[
\int_{\partial \Omega} e^{z \cdot x} \left[ (\bar{z} \cdot \nu) u_1 - \frac{\partial u_1}{\partial \nu} \right] d\sigma = \int_{\partial \Omega} e^{z \cdot x} \left[ (\bar{z} \cdot \nu) u_2 - \frac{\partial u_2}{\partial \nu} \right] d\sigma.
\]
Temporarily assume (4). Combining (3) and (4) we see
\[
\int_{\Omega} e^{z \cdot x} v_1 u_1 = \int_{\Omega} e^{z \cdot x} v_2 u_2.
\]
Since \( u_i = e^{iz\cdot x}m_{z,i} \), we get
\[
\int_{\Omega} e^{iz\cdot x}m_{z,1}v_1 = \int_{\Omega} e^{iz\cdot x}m_{z,2}v_2.
\]
Letting \( |z| \to \infty \) and using (c) of Lemma 1, we conclude
\[
\int_{\Omega} e^{iz\cdot x}v_1 = \int_{\Omega} e^{iz\cdot x}v_2.
\]
This shows \( v_1 = v_2 \). We now show (4). We apply Green’s theorem to the collar neighborhood \( D_\rho \). Since \( v_1 = v_2 = 0 \) in \( \mathbb{R}^n \setminus \Omega \), \( u_1 \) and \( u_2 \) are harmonic and \( C^\infty \) in \( \mathbb{R}^n \setminus \Omega \). By Lemma 2, \( u_1 = u_2 \) in \( \mathbb{R}^n \setminus \Omega \). Thus
\[
\int_{\partial D_\rho} e^{iz\cdot x} \left[ (\hat{z} \cdot \nu)u_1 - \frac{\partial u_1}{\partial \nu} \right] d\sigma = \int_{\partial D_\rho} e^{iz\cdot x} \left[ (\hat{z} \cdot \nu)u_2 - \frac{\partial u_2}{\partial \nu} \right] d\sigma.
\]
So by Green’s theorem
\[
\int_{\partial D_\rho} e^{iz\cdot x} \left[ (\hat{z} \cdot \nu)(u_1 - u_2) - \frac{\partial u_1}{\partial \nu} + \frac{\partial u_2}{\partial \nu} \right] d\sigma = \int_{D_\rho} e^{iz\cdot x}(v_1u_1 - v_2u_2).
\]
But \( \int_{D_\rho} |u_1| V < \infty \). Thus as \( \rho \to 0 \), the integral on the right side converges to zero. Thus the integral on the left side converges to zero. But in the limit the integral on the left side is exactly the difference of the two integrals in (4). This establishes (4).

We now give the argument by D. Jerison and C. Kenig. In essence we show that a form of Lemma 1 holds with no smallness assumption if \( v_i \in L^r(\Omega) \), \( r > n/2 \).

We begin with,

**Lemma 3.** Let \( 2/(n + 1) \leq (q - 2)/q \leq 2/n \). Let \( |z| = 1 \), and \( z \cdot z = 0 \). Then for \( 1/p + 1/q = 1 \),
\[
\|G_z f\|_{L^q(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)}
\]
where \( G_z f(\xi) = (|\xi|^2 + z \cdot \xi)^{-1} \hat{f}(\xi) \) and \( c \) is independent of \( f \) and \( z \).

**Proof.** We assume w.l.o.g. that
\[
G_z f(\xi) = (|\xi|^2 - 2\xi_1 + 2i\xi_2)^{-1} \hat{f}(\xi).
\]
By changing variables in \( \xi_1, \xi_1 \to (\xi_1 - 1) \) we can assume that \( G_z f(\xi) = (|\xi|^2 + 1 + 2i\xi_2)^{-1} \hat{f}(\xi) \). Since \( 1/p + 1/q = 1 \), \( (q - 2)/q = 1/p - 1/q \), and thus under the hypothesis of the lemma, \( 2/(n + 1) \leq 1/p - 1/q \leq 2/n \). We may thus apply Theorem 2.4 in [KRS] to conclude Lemma 3.

From Lemma 3 we deduce the next lemma. The notation we adopt is identical to Lemma 1.

**Lemma 4.** Let \( w_{z,i}(x) = m_{z,i}(x) - 1 \). Let \( v_i \in L^r(\Omega) \), \( r > n/2 \), and \( z \cdot z = 0 \). Let \( 2/(n + 1) \leq (q - 2)/q = 1/r < 2/n \). Then, for \( |z| \) large,

(a) there is a unique solution to \(-\Delta + v_i\) of the form \( u_i(x) = e^{iz\cdot x}m_{z,i}(x)\), with \( \|w_{z,i}\|_{L^q(\mathbb{R}^n)} \leq c \) uniformly in \( z \).

(b) \( \|w_{z,i}\|_{L^q(\mathbb{R}^n)} \to 0 \) as \( |z| \to \infty \).
Proof. From (1) we readily see that \( w_{z,i} \) satisfies
\[
 w_{z,i} + G_z(v_i w_{z,i}) = G_z(v_i \chi_\Omega) .
\]
Let \( M_{v_i}(f) = v_i f \), the multiplication by \( v_i \) operator. The identity above can be rewritten as,
\[
 (I + G_z M_{v_i})(w_{z,i}) = G_z M_{v_i}(\chi_\Omega) ,
\]
where \( I = \text{identity operator} \). We now claim that for \( \alpha = 2 - n/r > 0 \), and for \( c \) independent of \( z \),
\[
 \|G_z M_{v_i}(f)\|_{L^1(\mathbb{R}^n)} \leq c|z|^{-\alpha} \|f\|_{L^1(\mathbb{R}^n)} .
\]
Temporarily assume (6) and note that for large \( |z| \), \( I + G_z M_{v_i} \) is invertible on \( L^q(\mathbb{R}^n) \), and
\[
 \|G_z M_{v_i}(\chi_\Omega)\|_{L^q(\mathbb{R}^n)} \leq c|z|^{-\alpha}|\Omega|^{1/q} ,
\]
c independent of \( z \). Thus the uniqueness and existence of \( w_{z,i} \) follows from (5) and \( \|w_{z,i}\|_{L^s(\mathbb{R}^n)} \leq c|z|^{-\alpha} \). So we are reduced to checking (6). Let \( \delta = |z| \), and \( T_\delta f(x) = f(\delta x) \). We note \( G_z = \delta^{-2} T_\delta G_{z_\delta^{-1}} T_{\delta^{-1}} \), by a change of variables, and moreover \( \|T_\delta f\|_{L^q(\mathbb{R}^n)} = \delta^{-n/q} \|f\|_{L^q(\mathbb{R}^n)} \). Thus,
\[
 \|G_z M_{v_i}(f)\|_{L^q} = \delta^{-2-n/q} \|G_{z_\delta^{-1}} T_{\delta^{-1}} M_{v_i} f\|_{L^q} .
\]
By Lemma 3, the right side above is at most
\[
 c\delta^{-2-n/q} \|T_{\delta^{-1}} M_{v_i}(f)\|_{L^q} \leq c\delta^{-2-n/q+n/p} \|v_i f\|_{L^p} , \quad 1/p + 1/q = 1 .
\]
Now \( r^{-1} = 1 - 2q^{-1} = p^{-1} - q^{-1} \), because \( p^{-1} + q^{-1} = 1 \). So the right side above is at most \( c\delta^{-2+n/r} \|v_i f\|_{L^p} \). Now applying Holder’s inequality with exponents \( r/p \) and \( r/(r - p) = p/q \),
\[
 c\delta^{-2+n/r} \|v_i f\|_{L^p} \leq c\delta^{-2+n/r} \|v_i f\|_{L^q(\Omega)} \|f\|_{L^s(\mathbb{R}^n)} 
\leq |z|^{-\alpha} \|f\|_{L^s(\mathbb{R}^n)} .
\]
Thus we have (6), and Lemma 4 follows.

Using Lemma 4 we may conclude the fact that \( \Lambda_v \) determines \( v \) exactly as before.

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