DEFINING BERNOULLI POLYNOMIALS IN $\mathbb{Z}/p\mathbb{Z}$
(A GENERIC REGULARITY CONDITION)

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Abstract. We consider the problem of whether Bernoulli polynomials are uniquely defined by certain interpolation equations. This leads to an interesting characterization of regular primes, a new insight into the $p$-divisibility of Fermat quotients, and a generalization of Voronoi's congruences.

The Bernoulli polynomials $B_m(x)$, $m \in \{0, 1, 2, \ldots\} = \mathbb{N}$, satisfy the difference equation

$$F(x + 1) - F(x) = mx^{m-1}$$

and the interpolation equation

$$F(x) = q^{m-1} \left[ F\left(\frac{x}{q}\right) + F\left(\frac{x+1}{q}\right) + \cdots + F\left(\frac{x+q-1}{q}\right) \right]$$

for each integer $q \geq 2$, and for all real numbers $x$.

It is not hard to show that either (1) or (2) together with the value of $F(0)$ completely characterize the polynomial $F(x)$. However, Dickey, Kairies, and Shank [2] showed that in the field $\mathbb{Z}/p\mathbb{Z}$ this does not necessarily happen when $q = 2$ in (2). In this note we extend their work and give explicit criteria to determine when the above characterization occurs.

Throughout we shall assume that the prime $p$ is given and that we are considering the equations (1) and (2) only for functions $F: \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. We start by observing that, as a consequence of Fermat's Little Theorem ($x^p = x$ for all $x \in \mathbb{Z}/p\mathbb{Z}$), we need only consider values of $m$ in the range $1 \leq m \leq p - 1$.

Let $n$ be the least positive residue of $m$ (mod $p - 1$). In (1) if $p$ divides $m$ then $F(x) = F(0)$ for all $x$; otherwise $G(x) = F(x)/m$ satisfies $G(x + 1) - G(x) = x^{m-1} - x^{n-1}$. In (2) we simply can replace $q^{m-1}$ by $q^{n-1}$.

The Von Staudt–Clausen theorem [3,1] states that $p$ divides the denominator of the $m$th Bernoulli number $B_m$ exactly when $p - 1$ divides $m$ (and that $p$ divides $pB_{p-1} + 1$). Thus $B_m(x)$ is not well defined in $\mathbb{Z}/p\mathbb{Z}$ for $m \geq p - 1$. 

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We adjust for this problem by instead using the function

\[ C_m(x) = \frac{1}{m} [B_m(x) - B_m] \]

which, by (1), equals the sum of the \( m \)-th powers of the nonnegative integers less than \( x \), at each positive integer \( x \). Our key lemma is

**Lemma 1.** Suppose that \( p \) is a given prime and \( n \) and \( v \) are integers with \( 1 \leq n \leq p-1 \) and \( F(x+1) - F(x) = vx^{n-1} \) for each nonzero \( x \) in \( \mathbb{Z}/p\mathbb{Z} \). Then \( F(x) = vC_n(x) + F(0) \) in \( \mathbb{Z}/p\mathbb{Z} \).

**Proof.** Let \( G(x) = F(x) - vC_n(x) \). By (1) we have \( G(x+1) - G(x) = 0 \) for each \( x \neq 0 \). Therefore \( G(p) = G(p - 1) = \cdots = G(1) \) and \( G(p) = G(0) = F(0) \), giving the result.

Taking \( n \) to be the least positive residue of \( m \) (mod \( p \)) in Lemma 1 gives

**Theorem 1.** For any given prime \( p \) and integer \( m \), the equation (1) together with the value of \( F(0) \) completely (and uniquely) characterize a function \( F : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \).

We now move on to the more interesting (and difficult)

**Theorem 2.** Suppose that prime \( p \) and integers \( m \) and \( q \) are given, where \( 1 \leq m \leq p-1 \) and \( q \) is a primitive root (mod \( p \)). For any function \( F : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) we have that (2) is satisfied for every \( x \in \mathbb{Z}/p\mathbb{Z} \) if and only if

(a) \( F(x) = (F(2) - F(1))B_m(x)/m \) \quad for \( m \leq p - 2 \);

(b) \( F(x) = (F(2) - F(1))C_{p-1}(x) + F(0) \) \quad for \( m = p - 1 \),

where

\[ (F(2) - F(1)) \frac{q^{p-1} - 1}{p} = 0. \]

**Proof.** Let \( G(x) = F(x+1) - F(x) \) so that, by taking the difference of the equations for \( qx + 1 \) and \( qx \) in (2) we get

\[ G(qx) = qm^{-1}G(x) \quad \text{for each} \ x \in \mathbb{Z}/p\mathbb{Z}. \]

Therefore \( G(x) = vx^{m-1} \) by (3), where \( v = G(1) \), as \( q \) is a primitive root (mod \( p \)). By Lemma 1 this gives \( F(x) = vC_m(x) + F(0) \). When we substitute this back into (2) we find that

\[ (q^m - 1) \left( F(0) - \frac{v}{m}B_m \right) = 0. \]

(a) If \( 1 \leq m \leq p - 2 \) then \( p - 1 \) does not divide \( m \) and so, as \( q \) is a primitive root, \( (q^m - 1) \neq 0 \). Therefore \( F(0) = (v/m)B_m \) and so we get \( F(x) = vB_m(x)/m \).

(b) If \( m = p - 1 \) then, as \( F(0) \in \mathbb{Z}/p\mathbb{Z} \), we have \( (q^m - 1)F(0) = 0 \); thus, as \( pB_{p-1} \equiv p - 1 \) (mod \( p \)) (by the Von Staudt–Clausen theorem), we get from (4) that \( v(q^{p-1} - 1)/p = 0 \).
A number of corollaries follow.

**Corollary 1.** Suppose that prime \( p \) and integers \( m \) and \( q \) are given, where \( 1 \leq m \leq p - 1 \) and \( q \) is a primitive root \((\mod p)\). Then the equation (2), together with the value of \( F(0) \), completely (and uniquely) characterize the function \( F: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) if and only if \( (q^m - 1)B_m \neq 0 \ (\mod p) \).

**Proof.** In order for \( F(x) \) to be completely characterized we see, from Theorem 2, that we must be able to compute the value of \( F(2) - F(1) \) from the given information.

In (a) this occurs only when \( B_m(0)/m \neq 0 \ (\mod p) \) (in which case \( F(x) = F(0)B_m(x)/B_m \)), which is equivalent to \( (q^m - 1)B_m \neq 0 \ (\mod p) \), as \( (q^m - 1) \neq 0 \ (\mod p) \) and \( B_m = B_m(0) \).

In (b), as \( C_{p-1}(0) = 0 \), the value of \( F(2) - F(1) \) can be computed only if \( (q^{p-1} - 1)/p \neq 0 \ (\mod p) \) (in which case \( F(x) = F(0) \)) which is equivalent to \( (q^{p-1} - 1)B_{p-1} \neq 0 \ (\mod p) \) by the Von Staudt–Clausen theorem.

A prime \( p \) is defined to be regular if \( p \) does not divide the class number of the cyclotomic field \( K = \mathbb{Q}(\zeta_p) \), which means that for any ideal class \( \mathfrak{a} \) of \( K \), there exists an ideal class \( \mathfrak{d} \), for which \( \mathfrak{d}^p = \mathfrak{a} \). Kummer showed, as a consequence of this, that Fermat's last theorem is true for any regular prime exponent \( p \). He also proved that \( p \) is regular if and only if \( p \) does not divide any of the Bernoulli numbers \( B_m \) for \( m \) even with \( 1 \leq m \leq p - 3 \). We give here an equivalent set of criteria to regularity:

**Corollary 2.** A given prime \( p \) is regular if and only if for a given primitive root \( q \ (\mod p) \), equation (2) \( (\text{for each } x \in \mathbb{Z}/p\mathbb{Z}) \) together with the value of \( F(0) \) completely and uniquely characterize the function \( F: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) for each even positive integer \( m \), less than \( p - 2 \).

It should be noted that \( B_m = 0 \) for any odd integer \( m > 1 \), and so one can deduce from Corollary 1:

**Corollary 3.** Suppose that prime \( p \) and integers \( m \) and \( q \) are given, where \( 3 \leq m \leq p - 2 \), \( m \) is odd and \( q \) is a primitive root \((\mod p)\). Then the equation (2), together with the value of \( F(0) \), is satisfied by more than one function \( F: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \).

It is also important in number theory to study those values of \( q \) for which \( p \) divides the “Fermat quotient” \( (q^{p-1} - 1)/p \). Theorem 2 gives a new insight into that question:

**Corollary 4.** Suppose that prime \( p \) and integer \( q \) are given, where \( q \) is a primitive root \((\mod p)\). There exists a nonconstant function \( F: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) that satisfies

\[
F(x) = \frac{1}{q} \left[ F \left( \frac{x}{q} \right) + F \left( \frac{x+1}{q} \right) + \cdots + F \left( \frac{x+q-1}{q} \right) \right]
\]

if and only if \( p^2 \) divides \( q^{p-1} - 1 \).
Another interesting question is to consider (2) in the case that \( q \) is not a primitive root (mod \( p \)).

**Theorem 3.** Suppose that prime \( p \), and integers \( m, q, k \) are given, where \( 1 \leq m \leq p - 1, k \) divides \( p - 1 \) and \( q \) is of order \( (p - 1)/k \) (mod \( p \)). Suppose that the multiplicative subgroup of \( \mathbb{Z}/p\mathbb{Z} \setminus \{0\} \), generated by \( q \), has cosets \( A_1, A_2, \ldots, A_k \). For any function \( F: \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) we have that (2) is satisfied for every \( x \in \mathbb{Z}/p\mathbb{Z} \) if and only if there exist \( v_1, v_2, \ldots, v_k \in \mathbb{Z}/p\mathbb{Z} \) such that

\[
F(x) = F(1) + \sum_{i=1}^{k} v_i \sum_{1 \leq y \leq x-1 \atop y \in A_i} y^{m-1} \quad \text{for } 1 \leq x \leq p
\]

and

\[
q(q^{-m} - 1)F(1) = \sum_{i=1}^{k} v_i \sum_{1 \leq y \leq p-1 \atop y \in A_i} y^{m-1} \left[ \frac{qv}{p} \right].
\]

**Remark.** These formulae are, of course, natural generalizations of the famous formulae of Voronoi [4]: i.e. Take each \( v_i = m \) to get

\[
q(q^{-m} - 1){B_m} \equiv m \sum_{1 \leq y \leq p-1} y^{m-1} \left[ \frac{qv}{p} \right] \quad \text{(mod } p).\]

**Proof.** As in the proof of Theorem 2, we see that (3) holds where \( G(x) = F(x + 1) - F(x) \). Therefore, if \( x \in A_i \) then \( G(x) = x^{m-1}v_i \), for some fixed \( v_i \in \mathbb{Z}/p\mathbb{Z} \), giving (5). Substituting (5) into (2) for \( x = 0 \) gives (6), and it is easily verified that if (5) and (6) are satisfied then so is (2).

**References**


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