THE SEMISIMPLICITY PROBLEM FOR p-ADIC GROUP ALGEBRAS

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Abstract. For a prime $p$ let $\Omega = \Omega_p$ denote the completion of the algebraic closure of the field of $p$-adic numbers with $p$-adic valuation $| \cdot |$. Given a group $G$ consider the ring of formal sums

$$l_1(\Omega, G) = \left\{ \sum_{x \in G} \alpha_x x : \alpha_x \in \Omega, |\alpha_x| \to 0 \right\}.$$ 

Motivated by the study of group rings and the complex Banach algebras $l_1(C, G)$, we consider the problem of when this ring is semisimple (semiprimitive). Our main result is that for an Abelian group $G$, $l_1(\Omega, G)$ is semisimple if and only if $G$ does not contain a $C_p^{\infty}$ subgroup. We also prove that $l_1(\Omega, G)$ is semisimple if $G$ is a nilpotent $p'$-group, an ordered group, or a torsion-free solvable group. We use a mixture of algebraic and analytic methods.

I. Introduction

Throughout $p$ is a fixed prime, and all fields are contained in $\Omega = \Omega_p$, the completion of the algebraic closure of the field of $p$-adic numbers $Q_p$, and contain $Q_p$. If $k$ is such a complete field denote by $| \cdot | : k \to \mathbb{R}$ the non-archimedean extension of the $p$-adic valuation on $Q_p$. Given a group $G$ consider the ring of formal sums

$$l_1(k, G) = \left\{ \sum_{x \in G} \alpha_x x : \alpha_x \in k, |\alpha_x| \to 0 \right\}$$

where $|\alpha_x| \to 0$ means that for every $\varepsilon > 0$, only finitely many of the $\alpha_x$ satisfy $|\alpha_x| \geq \varepsilon$. We are interested in the problem of whether the ring, $l_1(k, G)$, is semisimple (sometimes called semiprimitive) or not. Our main results are that for Abelian groups $G$, $l_1(k, G)$ is semisimple if $G$ does not contain a $C_p^{\infty}$

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subgroup (§3), and that for a large class of solvable groups including all nilpotent $p'$-groups, $l_1(\Omega, G)$ is again semisimple (§4).

The algebra $l_1(Q_p, G)$ was studied in [10]. They consider the question of when $l_1(Q_p, G)$ is Noetherian, Artinian, or prime and ask when it is semiprime. Our work was also motivated by the well-known result that the Banach algebra over $C$

$$l_1(G) = \left\{ \sum_{x \in G} \alpha_x x : \alpha_x \in C, \sum |\alpha_x| < \infty \right\}$$

(here $|\cdot|$ denotes the usual absolute value on $C$) is always semisimple (cf. [8] or [13]). In contrast it is known that $l_1(\Omega, C_{p\infty})$ is not semisimple [4].

We conclude the paper with a list of some open problems.

II. General results

Define $\| \cdot \| : l_1(k, G) \to R$ as follows: if $f = \sum \alpha_x x \in l_1(k, G)$ then $\|f\| = \sup_{x \in G} |\alpha_x|$. Also, write $\text{supp}(f) = \{ x \in G : \alpha_x \neq 0 \}$. In the following lemma we collect a number of (very easy to prove) properties of $l_1(k, G)$:

2.1. Lemma. Let $k$ be a complete subfield of $\Omega$ containing $Q_p$ and let $G$ be a group. Then:

(i) If $f \in l_1(k, G)$ then $\text{supp}(f)$ is countable.

(ii) $\|f\| = \max_{x \in G} |\alpha_x|$ if $f = \sum \alpha_x x$, and we have:

- $\|f\| \geq 0$ for all $f$, and $\|f\| = 0$ if and only if $f = 0$;
- $\|f + g\| \leq \max\{\|f\|, \|g\|\}$ and $\|f + g\| = \|f\|$ if $\|g\| < \|f\|$;
- $\|fg\| \leq \|f\| \|g\|$.

In particular the topology defined by $\| \cdot \|$ turns $l_1(k, G)$ into an ultrametric topological ring which is complete.

(iii) The maximal right (or left ideals) of $l_1(k, G)$ are closed.

(iv) Let $\varphi : l_1(k, G) \to \Omega$ be a $k$-algebra homomorphism. Then $\varphi$ is continuous if and only if $|\varphi(f)| \leq \|f\|$ for all $f \in l_1(k, G)$.

(v) Let $\varphi : G \to \Omega$ be a $k$-algebra homomorphism where $|\varphi(x)| = 1$ for all $x \in G$. Then $\varphi$ has a unique continuous extension to $l_1(k, G)$.

Proof. (i) and (ii) are trivial. As for (iii), let $M$ be a maximal right ideal of $l_1(k, G)$. Then its closure $\overline{M}$ is also a right ideal. If $1 \in \overline{M}$ then we can find a sequence $\{f_n\}$ of elements of $M$ with $f_n \to 1$. But if $\|1 - f_n\| < 1$ then $f_n = 1 - (1 - f_n)$ is a unit (with inverse $\sum_{i=0}^{\infty} (1 - f_n)^i \in l_1(k, G)$ since $l_1(k, G)$ is complete). This is impossible, so $1 \notin \overline{M}$ and so $\overline{M} = M$.

(iv) Assume $\varphi$ is continuous and suppose there exists $x \in G$ with $|\varphi(x)| \neq 1$. Choose $\alpha \in k$ with $0 < |\alpha| < 1$. Replacing $x$ by a suitable power we may assume that $|\varphi(x)| > \alpha^{-1}$. Consider the element $f = \sum_{i=0}^{\infty} \alpha^i x^i \in l_1(k, G)$. If $f_n = \sum_{i=0}^{\infty} \alpha^i x^i$ then clearly $\|f - f_n\| \to 0$, while $|\varphi(f_n) - \varphi(f_{n-1})| = |\alpha^n \varphi(x)^n| \to \infty$, so the sequence $\{\varphi(f_n)\}$ is not convergent in $\Omega$. This contra-
dicts the continuity of $\varphi$. Thus $|\varphi(x)| = 1$ for all $x \in G$. Let $f = \sum_{x \in G} a_x x \in l_1(k, G)$. By continuity $\varphi(f) = \sum a_x \varphi(x)$, thus $|\varphi(f)| \leq \|f\|$. The other direction is trivial.

(v) It is easy to check that the unique continuous extension is given by $\varphi(\sum a_x x) = \sum a_x \varphi(x)$. \hfill \square

If $R$ is a ring we write $J(R)$ for the Jacobson radical of $R$. The following result enables us to concentrate on countable groups:

2.2. **Lemma.** If $k$ is a complete field and $G$ is a group, then

$$J(l_1(k, G)) \subseteq \bigcup_H J(l_1(k, H)),$$

where the union ranges over all countable subgroups $H$ of $G$.

**Proof.** Let $f \in J(l_1(k, G))$ and let $H = \langle \text{supp}(f) \rangle$. We claim that $f \in J(l_1(k, H))$. If $g \in l_1(k, H)$ then there exists $f' \in l_1(k, G)$ with $f'(1 - g f) = 1$. Let $T$ be a left transversal of $H$ to $G$ containing 1, and write $f = \sum_{t \in T} t f'_t$, where $f'_t \in l_1(k, H)$. It follows from $\sum_{t \in T} t f'_t (1 - g f) = 1$ that $f'_t (1 - g f) = 1$. This is for all $g \in l_1(k, H)$, so $f \in J(l_1(k, H))$, as required. \hfill \square

If $k$ is a field write $X_k$ for the class of all groups $G$ for which $l_1(k, G)$ is semisimple. Not much is known about the class-theoretic properties of $X_k$. As an example we have:

2.3. **Theorem.** Let $k$ be a complete field, and let $\{N_\lambda : \lambda \in \Lambda\}$ be a directed system of normal subgroups of $G$ such that every $G/N_\lambda \in X_k$. Then $G \in X_k$.

**Proof.** Let $0 \neq f \in l_1(k, G)$, and write $f = \sum_{i=1}^r \alpha_i x_i + g$ where $|\alpha_i| = \|f\|$ for all $i = 1, \ldots, r$ and $\|g\| < \|f\|$. By assumption there exists $\lambda \in \Lambda$ with $x_i x_j^{-1} \notin N_\lambda$ for $1 \leq i \neq j \leq r$. The mapping $\theta : l_1(k, G) \to l_1(k, G/N_\lambda)$ obtained by extending the natural homomorphism $G \to G/N_\lambda$ is easily seen to be a well-defined surjective continuous ring homomorphism. Now $\theta(f) = \sum \alpha_i \theta(x_i) + \theta(g)$, the elements $\theta(x_i)$ are distinct, and $\|\theta(g)\| \leq \|g\| < \|f\|$. Thus $\theta(f) \neq 0$, and since $J(l_1(k, G/N_\lambda)) = \{0\}$ this implies that $f \notin J(l_1(k, G))$, as required. \hfill \square

2.4. **Corollary.** If $G$ is residually finite then $G \in X_k$ for all $k$.

**Proof.** If $N \ntrianglelefteq G$ then $l_1(k, G/N)$ is simply the group ring of $G/N$ over $k$, which is semisimple by Maschke’s Theorem ([11], 2.4.2). \hfill \square

As another example we have

2.5. **Theorem.** Let $k$ be complete, with $\overline{k}$ the residue class field of $k$. If the group ring $\overline{k}G$ is semisimple and has no zero divisors then $G \in X_k$.
Proof. Suppose $J = \mathcal{J}(l_1(k, G)) \neq 0$, so it contains an element $g$ with $\|g\| = 1$. Choose any $f \in l_1(k, G)$ with $\|f\| = 1$. If $\|1 - fg\| < 1$ then $fg = 1 - (1 - fg)$ is invertible, which is impossible since $g \in J$. Let $u \in l_1(k, G)$ be such that $u(1 - fg) = 1$. We claim that $\|u\| = 1$. First, note that

$$1 = \|u(1 - fg)\| = \|u\||1 - fg\| = \|u\|.$$

Now set

$$R = \{y \in l_1(k, G) : \|y\| \leq 1\},$$

$$M = \{y \in l_1(k, G) : \|y\| < 1\}.$$

Then $M$ is an ideal of $R$ and $R/M \cong \overline{kG}$ (for the coefficients of the elements of $R$ belong to the valuation ring of $k$, and the homomorphism onto $\overline{k}$ is easily seen to extend to one of $R$ onto $\overline{kG}$ with kernel $M$). Let $\pi : R \to R/M$ denote the canonical map. If $\|u\| > 1$ choose $\alpha \in k$ with $|\alpha| = \|u\|$. Then $(\alpha^{-1}u)(1 - fg) = \alpha^{-1}$, and so $(\alpha^{-1}u)\pi(1 - fg)\pi = \alpha^{-1}\pi = 0$. But this is impossible since $\|\alpha^{-1}u\| = \|1 - fg\| = 1$ and $\overline{kG}$ has no zero divisors. Thus $\|u\| = 1$. But then $(\alpha u)(1 - f\pi g\pi) = 1$, and since this is true for all $f \in l_1(k, G)$ it follows that $g\pi \in J(\overline{kG}) = \{0\}$, contradicting $\|g\| = 1$.  

2.6. Corollary. Let $k$ be a complete field.

(i) If $G$ is a u.p. group (e.g. an ordered group) then $G \in X_k$.

(ii) If $k$ is uncountable and $kG$ has no zero divisors then $G \in X_k$.

Proof. These are consequences of 2.5 and well-known facts about the group-ring $\overline{kG}$ (cf. [11], 13.1.2, 13.1.9, 7.1.6) in view of the fact that u.p. groups are t.u.p. groups [14].

2.7. Corollary. Let $k$ be complete with residue class field $\overline{k}$, and let $G$ and $H$ be nontrivial groups. If the group ring $\overline{k}(G \times H)$ has no zero divisors then $G \times H \in X_k$.

Proof. The assumptions imply that $\overline{k}(G \times H)$ is semisimple [9]. The result follows from 2.5.

2.8. Corollary. Let $k$ be complete with residue class field $\overline{k}$. Suppose $G$ is a torsion-free solvable group. Then $G \in X_k$.

Proof. In [7] it is shown that for such groups $G$, $\overline{k}(G)$ has no zero divisors. It is well known that $\overline{k}(G)$ is semisimple (cf. [11] 7.4.6).

III. Abelian groups

For $f \in l_1(k, G)$ let $v(f) = \lim_{n \to \infty} \|f^n\|^{1/n}$ (it is easy to see that $v(f)$ is well defined, [12] 6.22). Say $f$ is topologically nilpotent if $v(f) = 0$. Our aim is to show that for Abelian groups $G$, the Jacobson radical of $l_1(k, G)$ is precisely the set of topological nilpotents in $l_1(k, G)$, as is the case for the
complex algebra \( l_1(G) \) [8]. We will write \( \Phi_k(G) \) for the set of all continuous \( k \)-algebra homomorphisms \( \varphi: l_1(k, G) \to \Omega \) and write \( \Phi'_k(G) \) for the set of \( \varphi \in \Phi_k(G) \) such that \( \varphi(l_1(k, G)) \) is a field. We have:

**3.1. Lemma.** Let \( G \) be an Abelian group, with \( H \) a subgroup of \( G \). Then any element of \( \Phi_k(H) \) (respectively, \( \Phi'_k(H) \)) can be extended to an element of \( \Phi_k(G) \) (respectively, \( \Phi'_k(G) \)).

**Proof.** Apply Zorn's Lemma to the set of all triplets \((G_1, \varphi, E)\) where \( G_1 \supseteq H \) is a subgroup of \( G \), \( \varphi \in \Phi_k(G_1) \) (respectively, \( \Phi'_k(G_1) \)), and \( E = \varphi(l_1(k, G_1)) \). It is routine to show that \( G \) belongs to a maximal element (cf. [11] 1.2.7).

We need to introduce an auxiliary seminorm on \( l_1(k, G) \): if \( f \in l_1(k, G) \) let

\[
\|f\|_{sp} = \sup \{|\varphi(f)| : \varphi \in \Phi'_k(G)\}.
\]

We have the following result:

**3.2. Lemma.** Let \( k \) be a complete field, and let \( G \) be an Abelian group.

(a) \( v(f + g) \leq \max\{v(f), v(g)\} \) for all \( f, g \in l_1(k, G) \).

(b) \( v(f) \leq \|f\| \) for all \( f \in l_1(k, G) \).

(c) \( v \) is continuous.

(d) \( \|f\|_{sp} \leq v(f) \leq p\|f\|_{sp} \) for all \( f \in l_1(k, G) \).

**Proof.** For (a) see [12], 6.22. Part (b) is obvious. Part (c) follows easily from (a) and (b). Consider (d). The inequality \( \|f\|_{sp} \leq v(f) \) follows from \( \|f^n\|_{sp} \leq \|f^n\| \). To prove the second inequality, first consider the case when \( G \) is finitely generated, and let \( \lambda \in k \) satisfy \( \|f\|_{sp} < |\lambda| \). Then \( |\varphi(\lambda^{-1}f)| = |\lambda^{-1}| \|\varphi(f)\| \leq |\lambda|^{-1} \|f\|_{sp} < 1 \) for all \( \varphi \in \Phi'_k \), and so \( (\lambda^{-1}f)^n \to 0 \) as \( n \to \infty \) [15]. Thus for all sufficiently large \( n \) we have \( \|\lambda^{-1}f^n\| < 1 \), whence \( \|f^n\|^{1/n} < |\lambda| \). Thus \( v(f) \leq |\lambda| \) whenever \( \lambda \in k \) satisfies \( \|f\|_{sp} < |\lambda| \). Since \( |k^*| \geq \{p^n : n \in Z\} \), and in particular has 0 as an accumulation point, it follows that \( v(f) = 0 \) if \( \|f\|_{sp} = 0 \). It also follows that if \( \|f\|_{sp} \neq 0 \) then there exists \( \lambda \in k \), with \( \|f\|_{sp} < |\lambda| \leq p\|f\|_{sp} \), and hence \( v(f) \leq p\|f\|_{sp} \).

In the general case write \( f = \lim_{n \to \infty} f_n \) where each \( f_n \in l_1(k, G) \) has finite support in \( G \). Given \( \varepsilon > 0 \) choose \( N \) large enough so that \( \|f - f_N\| < \varepsilon \), and \( |v(f_N) - v(f)| < \varepsilon \). Choose \( \varphi \in \Phi'_k(G) \) such that \( |\varphi(f_N)| > p^{-1} v(f_N) - \varepsilon \). (The existence of \( \varphi \) follows from the first part of the argument applied to \( \text{supp}(f_N) \)) and 3.1.) Thus

\[
\|f\|_{sp} \geq |\varphi(f)| \geq |\varphi(f_N)| - |\varphi(f - f_N)| > p^{-1} v(f_N) - \varepsilon - \|(f - f_N)\|
\]

\[
\geq p^{-1} v(f) - p^{-1} |v(f_N) - v(f)| - 2\varepsilon > p^{-1} v(f) - (2 + p^{-1})\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, the result follows. □
The proof shows that when \(|k^*|\) is dense in the positive real numbers, then we obtain \(v(f) \leq \|f\|_{sp}\), and so in fact \(v(f) = \|f\|_{sp}\).

We can now prove the following:

3.3. **Theorem.** Let \(k\) be a complete field, \(G\) an Abelian group, and \(f \in l_1(k,G)\). Then the following are equivalent:

(a) \(f \in J(l_1(k,G))\).

(b) \(\varphi(f) = 0\) for all \(\varphi \in \Phi_k(G)\).

(c) \(f\) is topologically nilpotent.

(d) \(\varphi(f) = 0\) for all \(\varphi \in \Phi_k(G)\).

**Proof.**

(a) \(\Rightarrow\) (b) is trivial, since \(\ker \varphi\) is a maximal ideal of \(l_1(k,G)\).

(b) \(\Rightarrow\) (c) follows from 3.2 since (b) implies that \(\|f\|_{sp} = 0\).

(c) \(\Rightarrow\) (a): let \(g \in l_1(k,G)\). Since \(\|g(f)^n\|^{1/n} \leq \|g\| \|f^n\|^{1/n} \to 0\) we have \(\|g(f)^n\| \to 0\), so \(\sum_{n=0}^{\infty} (g(f))^n \in l_1(k,G)\). In other words \(1 - gf\) is invertible, whence \(f \in J(l_1(k,G))\).

(c) \(\Rightarrow\) (d): a continuous homomorphism must map topological nilpotents to topological nilpotents, and the only topological nilpotent in a field is the zero element.

(d) \(\Rightarrow\) (b): Obvious. □

The following result is useful for dealing with extensions:

3.4. **Lemma.** Let \(G\) be Abelian, \(H\) a subgroup of \(G\), and let \(k\) be a complete field. Assume that \(H \in X_k\) and \(G/H \in X_E\) for all complete extension fields \(E \supseteq k\). Then \(G \in X_k\).

**Proof.** Let \(f = \sum_{x \in G} \alpha_x x \in J(l_1(k,G))\). Let \(\varphi \in \Phi_k'(H)\), and let \(\theta \in \Phi_k'(G)\) be any extension of \(\varphi\) (such exist by 3.1). Put \(E = \theta(l_1(k,G))\), a subfield of \(\Omega\). Let \(x \mapsto \overline{x}\) denote the natural map \(G \to G/H\). If \(\mu \in \Phi_E(G/H)\), where \(\overline{E}\) is the completion of \(E\), consider the map \(\psi: l_1(k,G) \to \overline{\Omega}\) defined by \(\psi(x) = \theta(x) \mu(\overline{x})\) for all \(x \in G\), and extended by linearity and continuity to the whole space. Since \(\psi \in \Phi_k(G)\), \(\psi(f) = 0\) by 3.3. Put \(\beta_x = \alpha_x \theta(x) \in E\), and \(\gamma_{\overline{x}} = \sum_{\overline{y} = \overline{x}} \beta_y \in E\). In view of the definition of \(\psi\) we have \(0 = \psi(f) = \sum_{\overline{x} \in G/H} \gamma_{\overline{x}} \mu(\overline{x}) = \mu(\sum_{\overline{x}} \gamma_{\overline{x}})\). This is for all \(\mu \in \Phi_E(G/H)\), and \(l_1(G/H)\) is semisimple by assumption, so by 3.3 again, \(\gamma_{\overline{x}} = 0\) for all \(\overline{x} \in G/H\). But \(\overline{y}\) is equivalent to \(y \in H x\) so we have \(\sum_{h \in H} \alpha_{hx} \theta(hx) = 0\), where \(x \in G\) is fixed. Cancelling a factor of \(\theta(x)\), and remembering that \(\theta\) is an extension of \(\varphi\), the above becomes \(\varphi(\sum_{h \in H} \alpha_{hx} h) = 0\). But this is for all \(\varphi \in \Phi_k'(H)\), so finally every \(\alpha_{hx} = 0\), as required. □

We need one more preliminary result.

3.5. **Lemma.** Let \(G\) be an Abelian \(p'\)-group, and let \(k\) be a complete subfield of \(\Omega\). Given distinct elements \(x_1, \ldots, x_n\) of \(G\) and elements \(c_1, \ldots, c_n\)
of $k$, there exists a continuous $k$-homomorphism $\varphi : l_1(k, G) \to \Omega$ such that $|\varphi(\sum_{i=1}^{n} c_i x_i)| = \|\sum_{i} c_i x_i\|$. 

Proof. Let $f = \sum c_i x_i$. We may assume that $\|f\| = 1$. If we can find a continuous $\varphi$ such that $|\varphi(f)| \geq 1$ then the trivial fact $|\varphi(f)| \leq \|f\|$ suffices to show that $|\varphi(f)| = 1 = \|f\|$.

The proof of $|\varphi(f)| \geq 1$ proceeds by induction on $n$, the case $n = 1$ being trivial (let $\varphi(x) = 1$ for all $x \in G$). Assume the result is true for $n - 1$ but false for $n$. Thus there exist distinct elements $x_1, \ldots, x_n \in G$, and $c_1, \ldots, c_n \in k$ with $\max_i |c_i| = 1$, such that $|\varphi(f)| < 1$ for all $\varphi \in \Phi_k(G)$ (where $f = \sum c_i x_i$).

We may suppose $|c_n| = 1$. It is easy to see that if $\lambda \neq 1$ is a $p'$-power root of unity in $\Omega$ then $|1 - \lambda| = 1$. Since $x_1^{-1} x_n \neq 1$ we can find a $p'$-power root of unity $\lambda \in \Omega$, $\lambda \neq 1$, of the same order as $x_1^{-1} x_n$. (If the order of $x_1^{-1} x_n$ is infinite let $\lambda \neq 1$ be any $p'$-power root of unity.) Define a continuous $k$-algebra homomorphism $\varphi : l_1(k, x_1^{-1} x_n) \to \Omega$ by $\varphi(x_1^{-1} x_n) = \lambda$. Extend this to an element (still denoted by $\varphi$) of $\Phi_k(G)$. Then

$$|\varphi(x_1) - \varphi(x_n)| = |\varphi(x_1)| |1 - \varphi(x_1^{-1} x_n)| = 1,$$

and so $\max_i |c_i (\varphi(x_1) - \varphi(x_i))| = 1$. By the inductive hypothesis there exists a continuous homomorphism $\mu : l_1(k, G) \to \Omega$ such that

$$(*) \quad \left| \sum_{i=1}^{n} c_i (\varphi(x_1) - \varphi(x_i)) \mu(x_i) \right| \geq 1.$$

The product homomorphism $\varphi \mu$ (defined by $\varphi \mu(\sum a_x x) = \sum a_x \varphi(x) \mu(x)$) is also continuous, and so

$$\left| \sum_{i=1}^{n} c_i \varphi(x_i) \mu(x_i) \right| < 1.$$

Since $|\varphi(x_1)| = 1$ we also have

$$\left| \sum_{i=1}^{n} c_i \varphi(x_i) \mu(x_i) \right| = \left| \sum_{i=1}^{n} c_i \mu(x_i) \right| < 1,$$

and thus

$$\left| \sum_{i=1}^{n} c_i (\varphi(x_1) - \varphi(x_i)) \mu(x_i) \right| < 1.$$

This contradicts $(*)$, and proves the result. □

Our main result on Abelian groups is

3.6. Theorem. Let $k$ be a complete subfield of $\Omega_p$, and let $G$ be an Abelian group with no $C_p^\infty$ subgroups. Then $G \in X_k$.

Proof. We proceed via a series of steps.

Step 1. We may assume that $G$ is a $p$-group: If $G_p$ denotes the maximal $p$-subgroup of $G$ and $E$ is a complete extension field of $k$, then we claim that
$G/G_p \in X_E$. For if $f \neq 0$ is an element of $l_1(E, G/G_p)$ then by 3.5 we can find a continuous homomorphism $\varphi \in \Phi_E(G/G_p)$ such that $||\varphi(f)|| = ||f|| \neq 0$, so $J(l_1(E, G/G_p)) = \{0\}$ by 3.3. Thus if $G_p \in X_k$, then $G \in X_k$ by 3.4. We are now in the case where $G$ is a countable (2.2) Abelian $p$-group with no $C_{p^\infty}$ subgroup.

Step 2. If $G$ has finite exponent $p^m$ then $G \in X_k$: If $G$ has exponent $p$ then it is a countable vector space over $GF(p)$, and is therefore residually finite. Thus $G \in X_k$ by 2.4. In general by induction on $m$ we have $G/G^{p^{m-1}}$ and $G^{p^{m-1}} \in X_k$ (for all fields $k$), and so $G \in X_k$ by 3.4.

Step 3. If $G$ has no element of infinite height then $G \in X_k$: The assumption is that $\bigcap_{m=1}^\infty G^{p^m} = \{1\}$, so $\{G^{p^m} : m = 1, 2, \ldots \}$ is a directed system in $G$. Each $G/G^{p^m} \in X_k$ by Step 2, and so $G \in X_k$ by 2.3.

We can now deal with countable reduced $p$-groups $G$. Consider the Ulm sequence of $G$ ([6], §76): put $G^{(1)} = \bigcap_{m=1}^\infty G^{p^m}$; if $\sigma$ is not a limit ordinal put $G^{(\sigma+1)} = G^{(\sigma)}(1)$, and if $\lambda$ is a limit ordinal put $G^{(\lambda)} = \bigcap_{\sigma<\lambda} G^{(\sigma)}$. Since $G$ is reduced we have $G^{(\tau)} = \{1\}$ for some ordinal $\tau$. We prove, by transfinite induction on $\sigma$, that $G/G^{(\sigma)} \in X_k$ for all $\sigma$. For $\sigma = 1$ the group $G/G^{(1)}$ has no elements of infinite height and so $G/G^{(1)} \in X_k$ by Step 3. If $\sigma$ is not a limit ordinal then $G^{(\sigma-1)}/G^{(\sigma)} \in X_k$ by Step 3, and $G/G^{(\sigma-1)} \in X_E$ (for all complete $E$) by induction, so $G/G^{(\sigma)} \in X_k$ by 3.4. If $\sigma$ is limit ordinal then $\{G^{(\rho)}/G^{(\sigma)} : \rho < \sigma\}$ is a directed system in $G/G^{(\sigma)}$, and since each $G/G^{(\rho)} \in X_k$ by induction, we have $G/G^{(\sigma)} \in X_k$ by 2.3. This establishes the inductive step. In particular $G = G/G^{(\tau)} \in X_k$, as required. \hfill $\Box$

In general $l_1(k, C_{p^\infty})$ is semisimple if and only if $k$ does not contain infinitely many $p$th-power roots of unity ([4]; see also [1], [5]). In fact, if $k^*$ contains a $C_{p^\infty}$ subgroup then $l_1(k, C_{p^\infty})$ even contains nonzero nilpotent elements. Thus no improvement to 3.6 without imposing additional restrictions on the field $k$ is possible and the following corollary is obvious:

3.7. Corollary. If $G$ is an Abelian group, then $l_1(\Omega, G)$ is semisimple if and only if $G$ does not contain a $C_{p^\infty}$ subgroup.

### IV. Solvable groups

To obtain results for solvable groups we need another extension theorem.

4.1. Lemma. Let $G$ be an Abelian $p'$-group and suppose $\{x_i\}_{i=1}^\infty \subset G$ with $x_i \neq x_j$ for $i \neq j$. Then there exists $\varphi_i \in \Phi_\Omega(G)$ such that if $A_n = (\varphi_i(x_j))_{i,j=1}^n$, then

1. $|\det A_n| = 1$ for all $n$, and
2. if $A_n^{-1} = (b_{ij}^{(n)})_{i,j=1}^n$, then $|b_{ij}^{(n)}| \leq 1$ for $i, j = 1, \ldots, n$. 

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Remark. This lemma can be compared to results on $k$-complete group algebras (c.f. [11], 4.3.3).

Proof. The proof proceeds by induction on $n$, the case $n = 1$ being trivial (take $\varphi_1 \equiv 1$). Assume the result holds for $n - 1$, so $\varphi_1, \ldots, \varphi_{n-1} \in \Phi_\Omega(G)$ have been found satisfying properties (1) and (2). Let $\mu \in \Phi_\Omega(G)$ and consider the $n \times n$ matrix $M(\mu) = (a_{ij})$ where

$$a_{ij} = \begin{cases} \varphi_i(x_j) & \text{if } i = 1, \ldots, n - 1, \ j = 1, \ldots, n \\ \mu(x_j) & \text{if } i = n, \ j = 1, \ldots, n. \end{cases}$$

Expanding the determinant along the $n$th row we obtain $\det M(\mu) = \sum_{i=1}^n \mu(x_i)c_i$, where $|c_i| \leq 1$ for $i = 1, \ldots, n - 1$ and $c_n = (-1)^{n-1} \det A_{n-1}$. By the induction assumption $|c_n| = 1$, thus by 3.5 there exists some $\varphi_n \in \Phi_\Omega(G)$ such that $|\sum_{i=1}^n c_i \varphi_n(x_i)| = 1$. Thus (1) is satisfied. Moreover the cofactor formula for inverses makes it clear that $|b_{ij}^{(n)}| \leq 1$ for $i, j = 1, \ldots, n$ establishing (2).

4.2. Theorem. Let $H$ be a normal subgroup of a group $G$ and suppose $G/H$ is an Abelian $p'$-group. If $f \in J(l_1(\Omega, G))$ then $f = \sum_{i=1}^\infty t_i x_i$ where $t_i \in J(l_1(\Omega, H))$, $x_i \in G$ and $\|t_i\| \to 0$.

Proof. Let $f \in J(l_1(\Omega, G))$ and suppose $f = \sum_{i=1}^\infty f_i$ with $f_i \in l_1(\Omega, G)$ and supp$f_i \subseteq Hx_i$ where $\{x_i\}_{i=1}^\infty$ are distinct coset representatives of $G/H$. Let $\pi: G \to G/H$ be the canonical quotient map. Apply 4.1 with the Abelian $p'$-group being $G/H$ and $\{x, \pi\}$ the distinct elements of $G/H$, to obtain $\varphi_i \in \Phi_\Omega(G/H)$ with the corresponding properties (1) and (2). Extend each $\varphi_i$ to elements of $\Phi_\Omega(G)$ by defining

$$\varphi_i \left[ \sum_{g \in G} a_g g \right] = \sum_{g \in G} a_g \varphi_i(g\pi).$$

Define

$$\varphi_i^* \left[ \sum_{g \in G} a_g g \right] = \sum_{g \in G} a_g \varphi_i(g)g.$$

Then $\varphi_i^*$ is an $\Omega$-automorphism of $l_1(\Omega, G)$. Notice that as supp$f_j \subset Hx_j$

$$\varphi_i^*(f) = \sum_{j=1}^\infty \varphi_i(x_j)f_j. \tag{*}$$

Let $\varepsilon > 0$ and choose $N$ such that $\|f_i\| < \varepsilon$ for $n > N$. Let

$$B_N = (\varphi_i(x_{N+j})) \quad \text{where } i = 1, \ldots, N \text{ and } j = 1, 2, \ldots.$$

From (*) we have the system of equations

$$(\varphi_i^*(f))_{i=1,\ldots,N} = A_N(f)_{i=1,\ldots,N} + B_N(f)_{i=N+1,N+2,\ldots}.$$

Hence

$$(f)_{i=1,\ldots,N} + A_N^{-1}B_N(f)_{i=N+1,N+2,\ldots} = A_N^{-1}(\varphi_i^*(f))_{i=1,\ldots,N}.$$
As $J(l_1(\Omega, G))$ is invariant under $\Omega$-automorphisms, \( \varphi_i^*(f) \in J(l_1(\Omega, G)) \). Let
\[
(j_i^{(N)})_{i=1, \ldots, N} = A_N^{-1}(\varphi_i^*(f))_{i=1, \ldots, N}.
\]
Then \( j_i^{(N)} \) is a linear combination of \( \varphi_1^*(f), \ldots, \varphi_N^*(f) \) and thus belongs to the Jacobson radical for each integer \( N \) and \( i = 1, \ldots, N \). Finally, let
\[
(e_i^{(N)})_{i=1, \ldots, N} = A_N^{-1}B_N(f_i)_{j=N+1, N+2, \ldots}.
\]
Then \( f_i = j_i^{(N)} + e_i^{(N)} \). As all the entries of \( A_N^{-1} \) and \( B_N \) have valuation at most one, (by the lemma) and \( \|f_n\| < \varepsilon \) for all \( n > N \), it follows that \( \|e_i^{(N)}\| < \varepsilon \) for all \( i = 1, \ldots, N \). As \( \varepsilon > 0 \) was arbitrary we have that, for each \( i \), \( \lim_{n \to \infty} \|j_i^{(N)} - f_i\| = 0 \) and as \( J(l_1(\Omega, G)) \) is closed (2.1(iii)) \( f_i \in J(l_1(\Omega, G)) \). Thus
\[
f_i x_i^{-1} \in J(l_1(\Omega, G)) \cap l_1(\Omega, H) \subseteq J(l_1(\Omega, H))
\]
(c.f. the proof of 2.2) and \( f = \sum (f_i x_i^{-1}) x_i \). \( \square \)

The next result is obvious.

4.3. Corollary. Suppose \( G_1 \) is an Abelian \( p' \)-group and \( G_2 \in X_\Omega \). Then \( G_1 \times G_2 \in X_\Omega \). \( \square \)

Let \( S_{p'} \) denote the class of solvable groups which have a subnormal series with Abelian \( p' \)-factor groups. This class contains all solvable torsion groups which have no elements of order \( p \) and all nilpotent \( p' \)-groups [2].

4.4. Theorem. Suppose \( H \) is a normal subgroup of \( G \) with \( H \in X_\Omega \) and \( G/H \in S_{p'} \). Then \( G \in X_\Omega \).

Proof. Let
\[
\{1\} = K_1 \leq K_2 \leq \cdots \leq K_N = G/H
\]
be a subnormal series for \( G/H \) with \( K_{i+1}/K_i \) a \( p' \)-Abelian group for each \( i = 1, \ldots, N \). Let \( \pi^{-1}(K_i) = G_i \). We have
\[
H = G_1 \leq G_2 \leq \cdots \leq G_n = G
\]
with \( G_{i+1}/G_i \cong K_{i+1}/K_i \) \( p' \)-Abelian groups, and as \( H \in X_\Omega \) an induction argument together with Theorem 4.2 now completes the proof. \( \square \)

4.5. Corollary. Let \( G \in S_{p'} \). Then \( G \in X_\Omega \). \( \square \)

4.6. Corollary. Let \( G \) have a directed system \( \{N_i : i \in I\} \) such that each \( G/N_i \in S_{p'} \). Then \( G \in X_\Omega \).

Proof. Combine Theorem 2.3 and Corollary 4.5. \( \square \)

4.7. Theorem. Let \( G \) have a directed system \( \{N_i : i \in I\} \) such that each factor group \( G/N_i \) is polycyclic. Then \( G \in X_\Omega \).

Proof. As usual we may assume \( G \) itself is polycyclic, so \( G \) has a subnormal series
\[
G = G_n \geq G_{n-1} \geq \cdots \geq G_0 = \{1\}
\]
with each $G_i/G_{i-1}$ a finitely generated Abelian group. The proof proceeds by induction on $n$. If $n = 1$ the result is clear, as finitely generated Abelian groups do not contain $C_p^\infty$. So assume $G_{i-1} \in X_\Omega$.

Choose a subgroup $L$ of $G$ with $G_i \geq L \geq G_{i-1}$, $G_i/L$ a torsion-free Abelian group and $L/G_{i-1}$ a finite Abelian group. Let $\pi: L \to L/G_{i-1}$ be the usual quotient map and assume $L/G_{i-1} = \{x_i\pi\}_{i=1}^N$, $x_i \in L$, with $x_i \pi$ distinct. By [11] 4.3.3 there exist homomorphisms $\varphi_1, \ldots, \varphi_N \in \Phi_{\Omega}(L/G_{i-1})$ such that the $N \times N$ matrix $(\varphi_i(x_j \pi))_{i,j=1}^N$ is nonsingular. Using this result in place of 4.1, arguments similar to those of the proof of 4.2, (but much easier as the matrix $B_N$ is unnecessary) show that if $f \in J(l_1(\Omega, L))$ then $f = \sum_{i=1}^N t_i x_i$ with $t_i \in J(l_1(\Omega, G_{i-1})) = (0)$. Thus $L \in X_\Omega$. As $G_i/L$ is an Abelian $p$-group, $G_i \in X_\Omega$ (4.2). This completes the induction step and hence the proof. □

Open problems

1. Is $l_1(Q_p, \oplus_{i=1}^N C_{p^\infty})$ semisimple? If so then it can be proved that $l_1(Q_p, G)$ is semisimple for any Abelian group $G$.

2. If $G$ is a nilpotent group which does not contain a $C_{p^\infty}$ subgroup, is $l_1(\Omega, G)$ semisimple?

3. Is the class $X_k$ closed under direct products or (normal) subgroups?

4. If $G \in X_k$ and $k$ is any complete extension field of $Q_p$ contained in $\Omega$, is $G \in X_k$?

References


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