RELATIVE LIE ALGEBRA COHOMOLOGY REVISITED

MARK MUZERE

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ABSTRACT. In this paper it is shown that relative Lie algebra cohomology is related to relative cohomology for restricted Lie algebras by a spectral sequence. Also an interpretation of the relative cohomology groups for Lie algebras in terms of the relative right derived functors of a certain relative derivative functor is given.

INTRODUCTION

Throughout this paper let (g, f) denote a pair of finite dimensional Lie algebras over a field k, where f is a subalgebra of g, and let (Ug, Uf) denote the corresponding pair of universal enveloping algebras of g and f. For a g-module W, let $H^n(Ug, Uf, W)$ denote the relative cohomology groups defined by $H^n(Ug, Uf, W) = \mathrm{Ext}^n_{Ug, Uf}(k, W)$, where $\mathrm{Ext}^n_{Ug, Uf}(k, W) = H^n(\mathrm{Hom}_{Ug}(X., W))$ and $X. \to k \to 0$ is a (Ug, Uf)-projective resolution of k [H].

In this paper we continue our study of the relative cohomology groups $H^n(Ug, Uf, W)$ started in [M]. (Also see [B-W, H] for related works.)

First if (g, f) is a pair of restricted Lie algebras, let (Vg, Vf) denote their restricted enveloping algebras (cf. §1). The reader is referred to [J] for more details. Define relative cohomology groups $H^n(Vg, Vf, W)$ for the pair of restricted Lie algebras (g, f) by $H^n(Vg, Vf, W) = \mathrm{Ext}^n_{Vg, Vf}(k, W)$, where W is a restricted g-module. More precisely,

$$\mathrm{Ext}^n_{Vg, Vf}(k, W) = H^n(\mathrm{Hom}_{Vg}(X., W)),$$

where $X. \to k \to 0$ is a (Vg, Vf)-projective resolution of k. The reader may consult [H] for further details. It is shown here that the relative cohomology groups $H^n(Ug, Uf, W)$ and $H^n(Vg, Vf, W)$ are related by the following spectral sequence.

(1.1) Let $0 \to W \to I^{\cdot}$ be a (Ug, Uf)-injective resolution of W.

Then there is a first quadrant spectral sequence

$$E_2^{i,j} = H^i(Vg, Vf, H^j(\mathrm{Hom}_{0(g)}(k, I^{\cdot}))) \Rightarrow H^{i+j}(Ug, Uf, W),$$

where $0(g)$ is as defined at the beginning of §2.

Now assume (g, f) to be a pair of not necessarily restricted Lie algebras. Let $\hat{H}^n(g, f, W)$ denote the relative right derived functors of a certain relative derivation functor (cf. §1). In [M], the group $H^1(Ug, Uf, W)$ has been identified with certain relative derivations modulo the inner ones. Here we provide a new interpretation of the higher relative cohomology groups $H^n(Ug, Uf, W)$. Namely,

$$(1.5) \qquad H^{n+1}(Ug, Uf, W) \cong \hat{H}^n(g, f, W) \quad \text{for } n \geq 1.$$

In §2, we construct the following Hochschild-Serre type spectral sequence.

(2.1) Let $f < h < g$ be a tower of Lie algebras where h is an ideal of g, $\bar{g} = g/h$, and let $0 \to W \to I^{\cdot}$ be a (Ug, Uf)-injective resolution of W. Then there is a first quadrant spectral sequence

$$E_2^{i,j} = H^i(U\bar{g}, H^j(\mathrm{Hom}_{Uh}(k, I^{\cdot}))) \Rightarrow H^{i+j}(Ug, Uf, W).$$

Moreover, if $f = 0$, we obtain the Hochschild-Serre spectral sequence

$$H^i(U\bar{g}, H^j(Uh, W)) \Rightarrow H^{i+j}(Ug, W).$$

Result (2.1) highlights certain subtleties in relative cohomology. The main observation here is that computation of the relative cohomology groups $H^n(Ug, Uf, W)$ is quite involved and in most cases a hopeless undertaking. These ideas will be made more precise in the appendix which deals with the analogous theory of relative group cohomology. An interpretation of the E_2-term of (3.1), a group cohomology analogue of (2.1) in terms of some familiar gadgets [K], turns out to be false.

1. COMPARISON THEOREMS

Unless stated otherwise throughout this section (g, f) will be a pair of restricted Lie algebras, and W will denote a restricted g-module. Define the restricted eveloping algebra of g by $Vg = Ug \otimes_{0(g)} k$, where $0(g)$ is the central subalgebra of Ug generated by $x^p - x^{[p]}, x \in g$. Here p is the characteristic of k. Let Vf be defined similarly. It is easy to see that there is a map $\varphi: H^*(Vg, Vf, W) \to H^*(Ug, Uf, W)$. Later we shall show that $H^1(Vg, Vf, W)$ is a subgroup of $H^1(Ug, Uf, W)$. First we show that the relative cohomology groups $H^n(Vg, Vf, W)$ and $H^n(Ug, Uf, W)$ are related by a spectral sequence. In fact, this is achieved by a generalization of [F-P, Proposition 5.3].

1.1. **Theorem.** *Let* $0 \to W \to I^{\cdot}$ *be a* (Ug, Uf)*-injective resolution of* W. *Then there is a first quadrant spectral sequence*

$$E_2^{i,j} = H^i(Vg, Vf, H^j(\mathrm{Hom}_{0(g)}(k, I^{\cdot}))) \Rightarrow H^{i+j}(Ug, Uf, W).$$

Proof. That $\text{Hom}_{0(g)}(k, I^{\cdot})$ is a complex of Vg-modules is clear. Let $J. \to k \to 0$ be a (Vg, Vf)-projective resolution of k. Now form the double complex $X_{ij} = \text{Hom}_{Vg}(J_i, \text{Hom}_{0(g)}(k, I^j))$ where $i, j \geq 0$. Following Cartan-Eilenberg [C-E] define $I_2 = H_I H_{II}(X)$ and $II_2 = H_{II} H_I(X)$. Thus

$$II_2^{i,j} = \begin{cases} H^j(Ug, Uf, W), & i = 0 \\ 0, & i > 0 \end{cases}$$

and $I_2^{i,j} = H^i(Vg, Vf, H^j(\text{Hom}_{0(g)}(k, I^{\cdot})))$. Therefore

$$E_2^{i,j} = H^i(Vg, Vf, H^j(\text{Hom}_{0(g)}(k, I^{\cdot}))) \Rightarrow H^{i+j}(Ug, Uf, W). \qquad \square$$

1.2. Corollary. $H^1(Vg, Vf, W)$ *is a subgroup of* $H^1(Ug, Uf, W)$.

Proof. Appeal to the standard five-term exact sequence $0 \to E_2^{1,0} \to H^1 \to \cdots$ associated to (1.1) $\square$

Recall the notation $W^f = \{w \in W | f.w = 0\}$. If we set $f = 0$ in the following result, we obtain [F-P, Proposition 5.3].

1.3. Corollary. *If f is an ideal of g, $\bar{g} = g/f$, we have*

$$E_2^{i,j} = H^i(V\bar{g}, H^j(0(\bar{g}), W^f)) \Rightarrow H^{i+j}(U\bar{g}, W^f).$$

Proof. This follows from (1.1) after making the following identifications. If we analyse the standard (Vg, Vf)-projective resolution of k, we obtain $H^*(Vg, Vf, W) \cong H^*(V\bar{g}, W^f)$. Since $H^*(Ug, Uf, W) \cong H^*(U\bar{g}, W^f)$, we may take the (Ug, Uf)-injective resolution of W in (1.1) to be an actual $U\bar{g}$-injective resolution of W^f. Therefore

$$H^*(\text{Hom}_{0(g)}(k, I^{\cdot})) \cong H^*(0(\bar{g}), W^f). \qquad \square$$

In what follows let (g, f) be a pair of not necessarily restricted Lie algebras. We now show that the higher relative cohomology groups $H^n(Ug, Uf, W)$ may be identified with the relative right derived functors of a relative derivation functor defined below. Recall that a linear map $h: g \to W$ such that $h([x, z]) = x \cdot h(z) - z \cdot h(x)$ for all $x, z \in g$ is called a derivation. Denote the set of all derivations from g to W that vanish on f by $\text{Der}(g, f, W)$. If f is an ideal of g, $\bar{g} = g/f$, then $\text{Der}(g, f,) \cong \text{Der}(\bar{g},)$. $\text{Der}(g, f,)$ is clearly a covariant left-exact functor from g-modules to k-vector spaces. Therefore it gives rise to certain relative Ext functors. These functors generalize certain well-known results in ordinary Lie algebra cohomology.

The following lemma is the key to the identification proposed above.

1.4. Lemma (Muzere [M]). *Let $Ig = \ker(\varepsilon: Ug \to k)$ and let If be defined similarly. Then, for any g-module W, the linear map*

$$\eta(\nu): \text{Hom}_{Ug}(Ig/UgIf, W) \to \text{Der}(g, f, W)$$

defined by

$$\eta(\nu)(h) = h \circ \nu \circ i,$$

where $i: g \to Ug$ *is the canonical embedding and* $h \circ \nu$ *is the extension of* $h\nu: Ig \to Ig/UgIf \to W$ *to* Ug *by zero, is a natural isomorphism.*

Let $\widehat{H}^n(g, f, W)$ denote the nth relative right derived functor of $\mathrm{Der}(g, f, \ )$ with coefficients in W. By Lemma 1.4,

$$\widehat{H}^n(g, f, W) \cong \mathrm{Ext}^n_{Ug, Uf}(Ig/UgIf, W)$$

for $n \geq 0$. Consequently $\widehat{H}^n(g, f, W)$ can be manipulated by using standard techniques in homological algebra. For example, since there are enough (Ug, Uf)-injectives, $\widehat{H}^n(g, f, \ )$ is coeffaceable for $n \geq 1$. In view of this, it is an easy matter to prove the following comparison theorem.

1.5. **Theorem.** $H^{n+1}(Ug, Uf, W) \cong \widehat{H}^n(g, f, W)$ *for* $n \geq 1$.

Proof. Since $Ig/UgIf \cong \ker(Ug \otimes_{Uf} k \to k)$, we have

$$H^{n+1}(Ug, Uf, W) \cong \mathrm{Ext}^n_{Ug, Uf}(Ig/UgIf, W) \quad \text{for } n \geq 1. \quad \square$$

If $f = 0$, we obtain the following result [Ba-R].

1.6. **Corollary.** $H^{n+1}(Ug, W) \cong \widehat{H}^n(g, W)$ *for* $n \geq 1$.

2. Hochschild-Serre type spectral sequence

In this section (g, f) will be a pair of not necessarily restricted Lie algebras and W will denote a g-module. In [M], it was pointed out that computations in relative cohomology seem to be very involved. The following result reinforces this point of view. §3 contains further elaboration on these matters.

2.1. **Theorem.** *Let* $f < h < g$ *be a tower of Lie algebras where* h *is an ideal of* g, $\bar{g} = g/h$, *and let* $0 \to W \to I^{\cdot}$ *be a* (Ug, Uf)-*injective resolution of* W. *Then there is a first quadrant spectral sequence*

$$E_2^{i,j} = H^i(U\bar{g}, H^j(\mathrm{Hom}_{Uh}(k, I^{\cdot}))) \Rightarrow H^{i+j}(Ug, Uf, W).$$

Moreover, if $f = 0$, *we obtain the Hochschild-Serre spectral sequence*

$$H^i(U\bar{g}, H^j(Uh, W)) \Rightarrow H^{i+j}(Ug, W).$$

Proof. Let $J. \to k \to 0$ be a $U\bar{g}$-projective resolution of k. Next form the double complex $X_{ij} = \mathrm{Hom}_{Ug}(J_i, \mathrm{Hom}_{Uh}(k, I^j))$, where $i, j \geq 0$, and proceed as in the proof of (1.1). $\square$

In the following result we extend the well known exact sequence in ordinary Lie algebra cohomology to the relative case.

2.2. Corollary. *Let* $f < h < g$ *be as in* (2.1). *Then there is an exact sequence*

$$0 \to H^1(Ug, Uh, W) \to H^1(Ug, Uf, W) \to E_2^{0,1}$$
$$\to H^2(Ug, Uh, W) \to H^2(Ug, Uf, W).$$

Proof. This follows from the standard five-term exact sequence associated to (2.1) and the fact that $H^n(Ug, Uh, W) \cong H^n(U\bar{g}, W^h)$ for $n \geq 0$. $\square$

We next show that under certain conditions the term $E_2^{0,1}$ has a nice interpretation.

2.3. Corollary. *Let the characteristic of* k *be* 0 *and let* $f < h < g$ *be as in* (2.1) *with* h *the radical of* g. *Then if* W *is finite dimensional,* $E_2^{0,1} \cong H^1(Ug, Uf, W)$.

Proof. Appeal to (2.2) and the two well-known Whitehead lemmas. $\square$

3. APPENDIX

Throughout this section G will denote a finite group. Let (G, H) be a pair of groups where H is a subgroup of G. For a (left) G-module A let $H^n(G, H, A)$ denote the relative cohomology groups defined by $H^n(G, H, A) = H^n(\mathrm{Hom}_G(X., A))$, where $X. \to Z \to 0$ is a (G, H)-projective resolution of Z [H].

In group cohomology, the cohomology groups of G are especially tractable if G has periodic cohomology [C-E] or if it admits a Hochschild-Serre spectral sequence argument. Extensions of these techniques to the relative case can be found in [Har], ([K],[S]) respectively. In this section we shall be concerned with the latter. In fact, we have

3.1. Theorem (Katamadze [K]). *If* $H < K < G$ *is a tower of groups where* K *is normal in* G, $\bar{G} = G/K$, *and if* $0 \to A \to I^{\cdot}$ *is a* (G, H)-*injective resolution of* A, *there is a first quadrant spectral sequence*

$$E_2^{i,j} = H^i(\bar{G}, H^j(\mathrm{Hom}_K(Z, I^{\cdot}))) \Rightarrow H^{i+j}(G, H, A).$$

Proof. Let $J. \to Z \to 0$ be a $\bar{G}$-projective resolution of Z. The result now follows from a standard argument applied to the double complex $X_{ij} = \mathrm{Hom}_{\bar{G}}(J_i, \mathrm{Hom}_K(Z, I^j))$, where $i, j \geq 0$. $\square$

This spectral sequence would facilitate computations in relative group cohomology if determination of the groups $H^j(\mathrm{Hom}_K(Z, I^{\cdot}))$ was plausible. We now show that there is a persistent difficulty in determining these groups. More precisely, we have

3.2. Lemma. *The interpretation* $H^j(\mathrm{Hom}_K(Z, I^{\cdot})) = H^j(K, H, A)$ *for* $j \geq 0$ *in* [K] *is false in general.*

Proof. To see this we produce a contradiction to the following well-known result.

3.3. **Theorem** (Blowers [B1]). *Let $k = F_p$ and G be the Sylow p-subgroup of the symmetric group on p^t elements, where $t \geq 2$, and let H be some appropriate nonnormal subgroup of G. Then the cohomology ring $H^*(G, H, k)$ is not finitely generated.*

Assume the interpretation given in (3.2) of the E_2-term of (3.1). Let k be as in (3.3), and let G be any p-group with a nonnormal subgroup H. By a standard fact in group theory there is a tower of subgroups $H_1 = H < H_2 < \cdots < H_{r-1} < H_r = G$ where, for each i, H_i is normal in H_{i+1} and $H_{i+1}/H_i \cong C_p$, a cyclic group of order p. If we take $K = H_i$ for appropriate i, (3.1) yields spectral sequences

$$(3.4) \qquad E_2^{s,t} = H^s(C_p, H^t(H_i, H, k)) \Rightarrow H^{s+t}(H_{i+1}, H, k),$$

where $2 \leq i \leq r - 1$.

Observe that if $i = 2$, we have our initial tower of groups $H < H_2 < H_3$. Thus

$$E_2^{s,t} = H^s(C_p, H^t(H_2, H, k))$$
$$\cong H^s(C_p, H^t(C_p, k)) \qquad \text{(by [A, Theorem 3.2]).}$$

Hence $H^*(H_3, H, k)$ is a finitely generated ring. Employing ideas of Evens [E] (see also [Be]) in an inductive argument, one obtains $H^*(G, H, k)$, a finitely generated ring. Now if we choose G and H as in (3.3), we obtain a contradiction as desired. □

3.5. **Lemma.** *If we take the ring of coefficients to be the rational integers Z, (3.2) also holds.*

Proof. Take $G = D_8 = gp\langle b, s | b^2 = e = s^4, b^{-1}sb = s^{-1}\rangle$, $K = \{e, s^2, sb, s^3b\}$, and $H = \{e, sb\}$. Then we obtain a tower of groups $H < K < G$ where K is normal in G. Observe that H is not normal in G. Since H is cyclic, we have an exact sequence

$$0 \to H^2(G, H, Z) \to H^2(G, Z) \to H^2(H, Z)$$

by [A, Theorem 7.3]. From Evens and Priddy [E-Pr] (see also [C-E, p. 250]) we get $H^2(G, Z) \cong Z_2^2$ and $H^2(G, H, Z) \cong Z_2$. Now the interpretation in (3.2) would yield $H^2(G, H, Z) \cong Z_2^2$ which is not consistent with the above calculation.   □

3.6. *Remark.* The argument in the proof of (3.5) also carries over to the tower of groups $H < K < G$, where $G = A_4$, $K = \{e, (12)(34), (13)(24), (14)(23)\}$ and $H = \{e, (12)(34)\}$.

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DEPARTMENT OF MATHEMATICS, CLARK UNIVERSITY, 950 MAIN STREET, WORCESTER, MASSACHUSETTS 01610-1477