PRODUCTS OF INFINITE-DIMENSIONAL SPACES

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Abstract. Observations concerning the product of R. Pol's weakly infinite-
dimensional uncountable-dimensional compactum with various spaces are
made. A proof showing that the product of a C-space and a compact C-
space is again a C-space is given. Related questions, motivated by this result,
are asked.

1. Introduction

By a space we mean a metric space and by the dimension of a space we mean
the Lebesgue covering dimension, for example as presented in [E1]. A space
is called countable-dimensional if it can be written as the union of a countable
number of finite-dimensional subspaces. If these subspaces can be chosen to be
closed in \( X \), then \( X \) is said to be strongly countable-dimensional. A space \( X \) is
weakly infinite-dimensional (in the sense of Alexandroff) if for every countable
family of pairs of disjoint closed subsets \( \{(A_n, B_n): n \in \mathbb{N}\} \) there exists a closed
separator \( S_n \subset X \) of each pair \( (A_n, B_n) \) so that the \( \bigcap\{S_n: n \in \mathbb{N}\} = \emptyset \). If a
space \( X \) is not weakly infinite-dimensional, then the space is said to be strongly
infinite-dimensional. An excellent survey of these topics is contained in [EP].

A space \( X \) has property C and is said to be a C-space if for any given
sequence of open covers \( \{\mathcal{U}_n: n \in \mathbb{N}\} \) there exists a refinement \( \mathcal{V}_n \) of \( \mathcal{U}_n \), for
each \( n \in \mathbb{N} \), so that each \( \mathcal{V}_n \) is a collection of pairwise disjoint open subsets of
\( X \); such a refinement is called a C-refinement, with the \( \bigcup\{\mathcal{V}_n: n \in \mathbb{N}\} \) forming
a cover of \( X \). This covering property was originally defined by Haver for metric
spaces and later generalized by Addis and Gresham to more general topological
spaces. Every countable-dimensional space has property C, and every space
with property C is weakly infinite-dimensional [H] [AG]. In particular, every
finite-dimensional space has property C.

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The following are well-known facts following directly from the definitions. More detail may be found in [G] and [AG].

**Subspace Theorems.** Every closed subspace of a weakly infinite-dimensional space (a C-space) is weakly infinite-dimensional (a C-space).

**Sum Theorems.** If a space \( X = \bigcup \{X_n: n \in \mathbb{N}\} \), where each subspace \( X_n \) is weakly infinite-dimensional (a C-space), then \( X \) is also weakly infinite-dimensional (a C-space).

**Local Theorems.** If every point of a space \( X \) has a neighborhood which is weakly infinite-dimensional (a C-space), then \( X \) is also weakly infinite-dimensional (a C-space).

It is not known whether every weakly infinite-dimensional space has property \( C \); however, R. Pol has constructed a weakly infinite-dimensional compactum containing a strongly infinite-dimensional subspace. Hence, this compactum cannot be countable-dimensional [P1]. When constructed as a subspace of the Hilbert cube, Pol’s compactum \( P \), which is known to have property \( C \), has the form \( P = X \cup Z \) where \( X \) is a topologically complete totally disconnected strongly infinite-dimensional subspace of the Hilbert cube with \( Z = P \setminus X \) countable-dimensional. Thus, there are disjoint Bernstein sets \( B_1 \) and \( B_2 \), i.e., all compact subsets of \( B_1 \) or \( B_2 \) are countable, so that \( Z = B_1 \cup B_2 \) [P2].

### 2. Examples

It is obvious from the subspace theorems that if \( X \) and \( Y \) are spaces whose product \( X \times Y \) is weakly infinite-dimensional (a C-space), then both factors \( X \) and \( Y \) must also be weakly infinite-dimensional (C-spaces).

The general converse to this statement is false. Indeed, using the notation given at the end of §1, it is known that \( B_1 \cup X \) and \( X \cup B_2 \) have property \( C \) [EP], and thus are weakly infinite-dimensional. However, the product \( (B_1 \cup X) \times (X \cup B_2) \) contains \( X \times X \) as a closed subspace, and so must be strongly infinite-dimensional [P2]. Neither factor could be compact, for if say \( B_1 \cup X \) were compact, then the projection to the other factor \( X \cup B_2 \) would be a closed mapping. This would make \( X \) a closed strongly infinite-dimensional subspace of \( X \cup B_2 \), contradicting the weak infinite-dimensionality of \( X \cup B_2 \). This example motivates the following question which we discuss in §3 of this paper.

**Question 1.** When is the product of two weakly infinite-dimensional spaces (C-spaces) again weakly infinite-dimensional (a C-space)?

It seems difficult to formulate positive product theorems for these infinite-dimensional dimension theories. Addis and Gresham have given the following product theorem for property \( C \) [AG].

**Theorem 1.** Let \( X \) and \( Y \) be two C-spaces. If \( Y \) is compact and has a basis \( \mathcal{B} \) of open sets such that for all \( B \in \mathcal{B} \) the product \( X \times \text{Bdy}(B) \) has property \( C \), then the product \( X \times Y \) also has property \( C \).
Corollary. If a space $X$ has property $C$ and $Y$ is a $\sigma$-compact strongly countable-dimensional space, then $X \times Y$ has property $C$.

Proof. Since the space $Y$ is $\sigma$-compact, $Y = \bigcup \{Y_n : n \in \mathbb{N}\}$ where each $Y_n$ is a compact. Moreover, since $Y$ is a strongly countable-dimensional space it is possible to write $Y = \bigcup \{Z_m : m \in \mathbb{N}\}$ where each $Z_m$ is a closed finite-dimensional subspace of $Y$. Since $Y_n = \bigcup \{Z_m \cap Y_n : m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$, it is seen that each compact subspace $Y_n$ is also strongly countable-dimensional. An inductive application of the theorem gives that each product $X \times (Z_m \cap Y_n)$ has property $C$, and thus we can apply the sum theorem once to $X \times Y_n = \bigcup \{X \times (Z_m \cap Y_n) : m \in \mathbb{N}\}$ and again to $X \times Y = \bigcup \{X \times Y_n : n \in \mathbb{N}\}$ to see that $X \times Y$ has property $C$. □

Y. Hattori has pointed out that $Y$ needs only to be a $\sigma$-compact countable-dimensional space, since a compact countable-dimensional space has small transfinite inductive dimension [E2, Corollary 4.7]. A simple argument using transfinite induction will then give the result. However, E. Pol has shown that the compactness of the countable-dimensional factor cannot be dropped [P].

Theorem 2. There exists a $C$-space whose product with a subspace of the irrationals is strongly infinite-dimensional, and thus cannot have property $C$.

3. Product theorems

In this section, we consider products of $C$-spaces, and, in particular, we consider the product of Pol's compactum with various weakly infinite-dimensional spaces.

Theorems 3. The product of a $C$-space with a compact $C$-space is again a $C$-space.

Proof. Let $X$ and $Y$ be $C$-spaces with $Y$ compact. Given a sequence of open covers of the product $X \times Y$, we rewrite the sequence as a countable collection $\{\mathcal{U}_{m,n} : n \in \mathbb{N}\}$ of sequences of open covers. Moreover, we may assume that each cover $\mathcal{U}_{m,n}$ is of the form

$$\mathcal{U}_{m,n} = \{A_{m,n}^\alpha \times B_{m,n}^\alpha : \alpha \in \Gamma_{m,n}\},$$

where each $A_{m,n}^\alpha$ is open in $X$ and every $B_{m,n}^\alpha$ is open in $Y$.

Fix $m \in \mathbb{N}$ and let $x \in X$ be fixed but arbitrary. For each $n \in \mathbb{N}$, we use the compactness of $Y$ to choose a finite subset $\Gamma_{m,n}(x)$ from the indexing set $\Gamma_{m,n}$ so that

$$\mathcal{B}_{m,n}(x) = \{B_{m,n}^\alpha : \alpha \in \Gamma_{m,n}(x)\}$$

is a finite cover of $Y$ with $x \in A_{m,n}^\alpha$ for each $\alpha \in \Gamma_{m,n}(x)$. Because $Y$ has property $C$, for each $n \in \mathbb{N}$ we can choose a $C$-refinement $\mathcal{D}_{m,n}(x)$ of $\mathcal{B}_{m,n}(x)$ so that the $\bigcup \{\mathcal{D}_{m,n}(x) : n \in \mathbb{N}\}$ is a cover of $Y$.  

We use the compactness of $Y$ again, this time to choose a positive integer $r_m(x) \in \mathbb{N}$ so that the $\bigcup \{D_{m,n}(x) : n = 1, \ldots, r_m(x)\}$ is a finite subcover of $Y$, and then we set

$$A_m(x) = \bigcap \{A_{m,n}^\alpha : n = 1, \ldots, r_m(x), \alpha \in \Gamma_{m,n}(x)\}.$$  

Since $\Gamma_{m,n}(x)$ is a finite set, $A_m(x)$ is an open neighborhood of $x$ in $X$. Thus, by constructing such an $A_m(x)$ for each $x \in X$ and defining

$$\mathcal{A}_m = \{A_m(x) : x \in X\},$$

we obtain an open cover of $X$.

In this manner, for each $m \in \mathbb{N}$ we construct such an open cover $\mathcal{A}_m$ of $X$. Since $X$ has property $C$, we can choose a $C$-refinement $\mathcal{C}_m$ of each $\mathcal{A}_m$ such that the $\bigcup \{\mathcal{C}_m : m \in \mathbb{N}\}$ covers $X$. Since each $\mathcal{C}_m$ is a refinement of $\mathcal{A}_m$, we can choose a function $\phi_m : \mathcal{C}_m \rightarrow X$ so that for each $C \in \mathcal{C}_m$ we have

$$C \subset A_m(\phi_m(C)),$$

and thus for any $n \in \{1, \ldots, r_m(\phi_m(C))\}$ and $\alpha \in \Gamma_{m,n}(\phi_m(C))$ we have

$$C \subset A_{m,n}^\alpha.$$

Finally, for each fixed $m, n \in \mathbb{N}$, we define

$$\mathcal{V}_{m,n} = \{C \times D : n \in \{1, \ldots, r_m(\phi_m(C))\} \text{ and } D \in D_{m,n}(\phi_m(C)) \text{ for some } C \in \mathcal{C}_m\}.$$

If $C \times D \in \mathcal{V}_{m,n}$, then $C \in \mathcal{C}_m$ with $D \in D_{m,n}(\phi_m(C))$, and thus $D \subset B_{m,n}^\alpha$ for some $\alpha \in \Gamma_{m,n}(\phi_m(C))$. Therefore, we see that

$$C \times D \subset A_{m,n}^\alpha \times B_{m,n}^\alpha \subset \mathcal{V}_{m,n},$$

so that $\mathcal{V}_{m,n}$ is an open refinement of $\mathcal{U}_{m,n}$. Moreover, since the elements of $\mathcal{C}_m$ are pairwise disjoint, and since for any fixed $C \in \mathcal{C}_m$ the elements of $D_{m,n}(\phi_m(C))$ are pairwise disjoint, we see that the elements of $\mathcal{V}_{m,n}$ are also pairwise disjoint.

To show that $X \times Y$ has property $C$, it only remains for us to show that $X \times Y$ is covered by the $\bigcup \{\mathcal{V}_{m,n} : m, n \in \mathbb{N}\}$. Let $(x, y)$ be an arbitrary point of $X \times Y$. Since the $\bigcup \{\mathcal{C}_m : m \in \mathbb{N}\}$ covers $X$, we can find $m \in \mathbb{N}$ and $C \in \mathcal{C}_m$ with $x \in C$. Since $Y$ is covered by the $\bigcup \{D_{m,n}(\phi_m(C)) : n = 1, \ldots, r_m(\phi_m(C))\}$, we can find $n \in \{1, \ldots, r_m(\phi_m(C))\}$ and $D \in D_{m,n}(\phi_m(C))$ so that $y \in D$. Hence,

$$(x, y) \in C \times D \in \mathcal{V}_{m,n},$$

which completes the proof. $\square$

**Corollary.** The product of Pol's compactum with any $C$-space has property $C$, and thus is weakly infinite-dimensional.
Corollary. For every $n \in \mathbb{N}$, the $n$-fold product $P^n$ of R. Pol's compactum has property $C$.

Corollary. The product of a $\sigma$-compact (locally compact) $C$-space with a $C$-space is again a $C$-space.

This last corollary follows from the sum and local theorems for property $C$.

Question 2. Let $f : X \to Y$ be an open and closed mapping between spaces $X$ and $Y$, where $Y$ is a $C$-space. If $f^{-1}(y)$ is a $C$-space for each $y \in Y$, then must $X$ also be a $C$-space?

References


