NEUMANN EIGENVALUE ESTIMATE
ON A COMPACT RIEMANNIAN MANIFOLD

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Abstract. In their article, P. Li and S. T. Yau give a lower bound of the first Neumann eigenvalue in terms of geometrical invariants for a compact Riemannian manifold with convex boundary. The purpose of this paper is to generalize their result to a compact Riemannian manifold with possibly nonconvex boundary.

1. Introduction

Let $M^n$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$. In local coordinates $(x^1, x^2, \ldots, x^n)$, the Riemannian metric is given by

\begin{equation}
    ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j.
\end{equation}

One defines on $M$ a second order elliptic differential operator by

\begin{equation}
    \Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),
\end{equation}

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$, which is known as the Laplace operator. The purpose of this paper is the study of eigenvalues of the Laplace operator. More specifically, we study the following problem:

Assume that $\partial M \neq \emptyset$, we adopt an “interior rolling $\varepsilon$-ball” condition on $\partial M$ to consider the following Neumann eigenvalue problem on $M^n$:

\begin{equation}
    \begin{cases}
        \Delta h = -\eta h, \\
        \frac{\partial h}{\partial \nu} \equiv 0 \text{ on } \partial M,
    \end{cases}
\end{equation}

where $\nu$ is the unit outward normal vector to the boundary $\partial M$.

Definition 1.1. Let $\partial M$ be the boundary of a compact Riemannian manifold $M^n$. Then $\partial M$ satisfies the “interior rolling $\varepsilon$-ball” condition if for each point...
There is a geodesic ball $B_q(\varepsilon/2)$, centered at $q \in M$ with radius $\varepsilon/2$, such that
\[ p = B_q(\varepsilon/2) \cap \partial M \quad \text{and} \quad B_q(\varepsilon/2) \subset M. \]

It is well known that the set of eigenvalues $\{\eta_k\}$ of (1.3) are nonnegative and can be arranged in a nondecreasing order of magnitude as follows:
\[ 0 = \eta_0 < \eta_1 \leq \eta_3 \leq \cdots \leq \eta_m \leq \cdots. \]

By the compactness of $M^n$, it is known that those functions which satisfy (1.3) with eigenvalue $\eta_0$ are constants. The first nonzero eigenvalue $\eta_1$ in the problem (1.3) is hence characterized as the optimal constant in the Poincaré inequality:
\[ \eta_1 \int f^2 \leq \int |\nabla f|^2 \]
for all $f \in H^2_1(M)$ such that $\int_M f = 0$. Due to the importance of Poincaré inequality for analysis on manifolds, one wishes to obtain optimal quantitative estimates for the first eigenvalue $\eta_1$ from below in terms of geometric elements. Classically, for domains in $\mathbb{R}^n$, lower estimates for $\eta_1$ were established by Payne-Weinberger [4] and Payne-Stakgold [3], etc. For general compact manifolds with convex boundary, the lower estimates of $\eta_1$ were obtained by Li-Yau [2]. Using a method similar to that of Li-Yau [2], we have the following:

**Theorem 1.1.** Let $M^n$ be a compact Riemannian manifold with boundary $\partial M$. Let $\partial M$ satisfy the "interior rolling $\varepsilon$-ball" condition. Let $K$ and $H$ be nonnegative constants such that the Ricci curvature $\text{Ric}_M$ of $M$ is bounded below by $-K$ and the second fundamental form elements of $\partial M$ is bounded below by $-H$. By choosing $\varepsilon$ "small", we have

\[ \eta_1 \left( 1 + \frac{\alpha^2}{(n-1)d^2} \right) \exp(-B) \leq \eta_1 \]

where $\alpha$ and $\varepsilon$ are positive constants less than 1.

\[ d = \text{diameter of } M^n, \]
\[ B = 1 + \left[ 1 + \frac{4(n-1)d^2C}{1 - \alpha^2} \right]^{1/2}, \]
\[ C = (1 + H)C_1 + \frac{[(2n-3)^2 + (4n-5)\alpha^2]H^2}{(n-1)\varepsilon^2 \alpha^2} + (H + 1)^2 K, \]

and
\[ C_1 = \frac{2(n-1)H(3H + 1)(H + 1)}{\varepsilon} + \frac{H + H^2}{\varepsilon^2}. \]

**Remark 1.** When the boundary $\partial M$ is convex, our estimate implies the estimate, obtained by Li-Yau [2, Theorem 9].
Remark 2. In our estimate, the choice of $\varepsilon$ depends on the upper bound of the sectional curvature near the boundary. We do not know whether we can determine the upper bound of $\varepsilon$ without using the curvature bound near the boundary. The upper bound of $\varepsilon$ is given by (2.16) and (2.17).

In §2, we shall give a gradient estimate which is essential in a proof of the main result. In §4, we shall give a proof of Theorem 1.1. In §3, a counterexample is given which will demonstrate that the “interior rolling $\varepsilon$-ball” condition is definitely necessary for $\eta_1$ being bounded away from 0.

2. A GRADIENT ESTIMATE

Throughout this section, $M^n$ is assumed to be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$. Let $K, H$ be nonnegative constants which were defined in §1. Let $f$ be a function which satisfies (1.3) with $\eta = \eta_1$; i.e., $f$ is an eigenfunction of (1.3) with eigenvalue $\eta_1$.

In this section, our goal is the study of the solution of equation (1.3) using maximal principle.

Let us first recall some general facts concerning a Riemannian manifold. Let $\{e_i\}$ be a local frame field of a Riemannian manifold $M^n$ and $\{\omega^i\}$ be the corresponding dual frame field. Then the structure equations of $M^n$ are given by

\begin{align}
(2.1) \quad & d\omega_j = \sum_{i=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji}, \\
(2.2) \quad & d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_{ij} \wedge \omega_k.
\end{align}

For any $C^2$-function $p(x)$ defined on $M^n$, we may define its gradient and Hessian by the following formulas:

\begin{align}
(2.3) \quad & dp = \sum_{i=1}^{n} p_i \omega_i, \\
(2.4) \quad & \sum_{j=1}^{n} p_j \omega_j = dp + \sum_{j=1}^{n} p_j \omega_{ji};
\end{align}

and the covariant derivatives of $p_{ij}$ are defined by

\begin{align}
(2.5) \quad & \sum_{k=1}^{n} p_{ijk} \omega_k = dp_{ij} + \sum_{k=1}^{n} p_{kj} \omega_{ki} + \sum_{k=1}^{n} p_{ik} \omega_{kj}.
\end{align}

By exteriorly differentiating (2.4), we get the following commutational formula:

\begin{align}
(2.6) \quad & p_{ijk} - p_{ikj} = \sum_{j=1}^{n} p_l R_{lijk}.
\end{align}
Theorem 2.1. Let $M^n$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$ satisfying the "interior rolling $\varepsilon$-ball" condition. Let $f$ be a solution of equation (1.4) with $\eta = \eta_1$. If $\mu > 1$ is any constant, and $\varepsilon$ is "small", then

\[
\frac{\|\nabla f\|^2}{(\mu \sup f - f)^2} \leq \max \left\{ \frac{4(n-1)}{1 - \alpha^2} \left[ C + (H + 1)^2 \eta_1 \left\| \frac{\mu \sup f}{\mu \sup f - f} \right\|_\infty \right] , \right.
\]

\[
\frac{\sqrt{8}(H + 1)^2}{\sqrt{1 - \alpha^2}} \eta_1 \left\| \frac{f}{\mu \sup f - f} \right\|_\infty \right\} ,
\]

where $\alpha$ and $\varepsilon$ are positive constants less than 1, and

\[
C = (1 + H)C_1 + \frac{[(2n - 3)^2 + (4n - 5)\alpha^2]H^2}{(n - 1)\varepsilon^2 \alpha^2} + (1 + H)^2 K ,
\]

\[
C_1 = \frac{2(n - 1)H(3H + 1)(1 + H)}{\varepsilon} + \frac{H + H^2}{\varepsilon^2} .
\]

Remark 1. When the boundary $\partial M$ is convex, our theorem implies the gradient estimate obtained in Li-Yau [2, Theorem 3].

Remark 2. In our estimate, $\varepsilon$ is chosen to be a positive constant less than 1 and is dependent on the upper bound of the sectional curvature of the manifold near the boundary. The upper bound of $\varepsilon$ is given by (2.16) and (2.17).

Proof. Let $\psi(r)$ be a nonnegative $C^2$ function defined on $[0, \infty)$ such that

\[
\psi(r) = \begin{cases} 
H & \text{if } r \in [0, \frac{1}{2}) , \\
H & \text{if } r \in [1, \infty) , 
\end{cases}
\]

with $\psi(0) = 0$, $2H \geq \psi'(r) \geq 0$, $\psi'(0) = H$

and $\psi''(r) \geq -H$.

Let

\[
\phi(x) = \psi \left( \frac{r(x)}{\varepsilon} \right) ,
\]

where $r(x)$ denotes distance between boundary $\partial M$ and $x \in M$. For $\mu > 1$, we define the function

\[
G(x) = (1 + \phi)^2 \frac{\|\nabla f\|^2}{(\mu \sup f - f)^2} .
\]

By the compactness of $M$, there is a point $p \in M$ such that $G$ achieves its supremum. We may assume that $G(p) > 0$, or else the theorem follows trivially. Suppose that $p$ is a boundary point of $\partial M$. At $p$ be may choose an
orthonormal frame \( e_1, \ldots, e_n \) such that \( e_n = \frac{\partial}{\partial \nu} \), where \( \nu \) is the unit outward normal vector to \( \partial M \). Then we have

\[
0 \leq \frac{\partial G}{\partial \nu}(p).
\]

This gives

\[
0 \leq \frac{\partial \phi}{\partial \nu} \frac{1}{1 + \phi} + \sum_{i=1}^{n} f_i f_{iv} - \frac{f \nu}{|\nabla f|^2} \mu \sup f - f
\]

(2.10)

\[
= -\frac{H}{\varepsilon} + \sum_{i=1}^{n-1} f_i f_{iv} \frac{|\nu/v|^2}{|\nabla f|^2}.
\]

If \( h_{ij} \) are the second fundamental form elements of \( \partial M \), then by a direct computation one shows that

\[
f_{iv} = -\sum_{j=1}^{n-1} h_{ij} f_j \quad \text{for } 1 \leq i \leq n - 1,
\]

(2.11)

where we used the fact that \( f \nu = 0 \) on \( \partial M \). Together with (2.9), we have

\[
0 \leq -\frac{H}{\varepsilon} - \frac{\sum_{i,j=1}^{n-1} h_{ij} f_i f_j}{|\nabla f|^2}
\]

\[
\leq -\frac{H}{\varepsilon} + H
\]

\[
< 0,
\]

which is a contradiction, as we choose \( \varepsilon \) to be smaller than 1. Hence \( G(x) \) cannot attain its maximum at the boundary point. Therefore \( p \) has to be an interior point of \( M \). Hence at \( p \)

\[
\nabla G = 0
\]

(2.12)

and

\[
\Delta G \leq 0.
\]

(2.13)

This gives

\[
0 = \frac{\psi' r_j}{\varepsilon(1 + \phi)} + \sum_{i=1}^{n} f_i f_{ij} + \frac{f_j}{\mu \sup f - f}
\]

(2.14)

and

\[
0 \geq \frac{\Delta \phi}{1 + \phi} - \frac{(\psi')^2}{\varepsilon^2 (1 + \phi)^2} + \left( \sum_{i,j=1}^{n} f_{ij}^2 + \sum_{i,j=1}^{n} f_i f_{ij} \right)/|\nabla f|^2
\]

\[
- \frac{2 \sum_{j=1}^{n} (\sum_{i=1}^{n} f_i f_{ij})^2}{|\nabla f|^4} + \frac{\Delta f}{\mu \sup f - f} + \frac{|\nabla f|^2}{(\mu \sup f - f)^2}.
\]

(2.15)
To compute $\Delta \phi$, we let $\partial M(e) = \{ x \in M | r(x) \leq e \}$ and $K_e$ be the upper bound of the sectional curvature in $\partial M(e)$. We may choose $e$ to be small so that

\[
\sqrt{K_e} \tan \left( e \sqrt{K_e} \right) \leq \frac{H}{2} + \frac{1}{2}.
\]

and

\[
\frac{H}{\sqrt{K_e}} \tan \left( e \sqrt{K_e} \right) \leq \frac{1}{2}.
\]

By using an index comparison theorem in Riemannian geometry [5, p. 347], one can show that if $x \in \partial M(e)$, we have

\[
\Delta r \geq -(n - 1) \frac{eH + e\sqrt{K_e} \tan(e \sqrt{K_e})}{e - e \frac{H}{\sqrt{K_e}} \tan(e \sqrt{K_e})}
\]

\[
\geq -(n - 1)(3H + 1).
\]

Then we have

\[
\Delta \phi = \frac{1}{e} \psi' \Delta r + \frac{1}{e^2} \psi'' |\nabla r|^2
\]

\[
\geq -\frac{2(n - 1)H(3H + 1)}{e} - \frac{H}{e^2}
\]

\[
= -C_1.
\]

At $p$, we may choose an orthonormal frame $\{e_i\}$ such that $f_j(p) = |\nabla f|(p)$. By using (2.6) and (2.14), we also have, at $p$,

\[
f_{j_1 j} - f_{j j_1} = \sum_{i=1}^n f_i R_{i j_1 j}
\]

and

\[
f_{i j} = \frac{\psi' f_{i j}}{e(1 + \phi)} - \frac{f_i f_j}{\mu \sup f - f}.
\]

Substituting these and (2.18) into (2.15), we have

\[
0 \geq -\frac{C_1}{1 + \phi} - \frac{[1 + r_1^2](\psi')^2}{e^2(1 + \phi)^2} - \frac{2\psi' f_{i j} r_1}{\mu \sup f - f}
\]

\[
+ \sum_{i > 1} \frac{f_{i i}^2}{f_{i i}^2} + Ric_{11} - \frac{\eta_1 \mu \sup f}{\mu \sup f - f},
\]

\[
\geq -\frac{C_1}{1 + \phi} - \frac{2(\psi')^2}{e^2(1 + \phi)^2} - \frac{2\psi' f_{i j} r_1}{\mu \sup f - f}
\]

\[
+ \sum_{i > 1} \frac{f_{i i}^2}{f_{i i}^2} + Ric_{11} - \frac{\eta_1 \mu \sup f}{\mu \sup f - f}.
\]
It is also clear that

\[ \sum_{i>1} f_{ii}^2 \geq \frac{1}{n-1} \left( \sum_{i>1} f_{ii} \right)^2 \]

(2.20)

\[ \geq \frac{f_{11}^2}{2(n-1)} - \frac{(\Delta f)^2}{n-1}. \]

Then we have

\[
\sum_{i>1} f_{ii}^2 \geq \frac{f_1^4}{2(n-1)(\mu \sup f - f)^2} + \frac{f_1^3 \psi' r_1}{\varepsilon(n-1)(1+\phi)(\mu \sup f - f)} \\
\quad \quad + \frac{4(n-1)(\psi')^2 - (\psi')^2 r_1^2}{2\varepsilon^2(n-1)(1+\phi)^2} - K - \frac{\eta_1^2 f^2}{\mu \sup f - f} - \frac{\eta_1^2 r_1^2}{(n-1)f_1^2}.
\]

(2.21)

Substituting (2.21) into (2.19), we have

\[
0 \geq \frac{f_1^2}{2(n-1)(\mu \sup f - f)^2} - \frac{(2n-3)f_1 \psi' r_1}{\varepsilon(n-1)(1+\phi)(\mu \sup f - f)} - \frac{C_1}{1+\phi} \\
\quad \quad - \frac{4(n-1)(\psi')^2 - (\psi')^2 r_1^2}{2\varepsilon^2(n-1)(1+\phi)^2} - K - \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 f^2}{(n-1)f_1^2}.
\]

(2.22)

It is clear that

\[
\frac{\alpha^2 f_1^2}{2(n-1)(\mu \sup f - f)^2} - \frac{(2n-3)f_1 \psi' r_1}{\varepsilon(n-1)(1+\phi)(\mu \sup f - f)} \\
\quad \quad \geq - \frac{(2n-3)^2(\psi')^2(r_1)^2}{2\varepsilon^2 \alpha^2(n-1)(1+\phi)^2},
\]

\[
\geq - \frac{(2n-3)^2(\psi')^2}{2\varepsilon^2 \alpha^2(n-1)(1+\phi)^2}.
\]

Substituting this into (2.22), we have

\[
0 \geq \left( \frac{1 - \alpha^2}{2(n-1)} \right) \frac{f_1^2}{(\mu \sup f - f)^2} + \frac{\alpha^2 - (2n-3)^2(\psi')^2(r_1)^2}{2\varepsilon^2 \alpha^2(n-1)(1+\phi)^2} \\
\quad \quad - \frac{2(\psi')^2}{\varepsilon^2(1+\phi)^2} - \frac{C_1}{1+\phi} - K - \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 f^2}{(n-1)f_1^2},
\]

(2.23)

\[
\geq \left( \frac{1 - \alpha^2}{2(n-1)} \right) \frac{f_1^2}{(\mu \sup f - f)^2} - \frac{\alpha^2 - (2n-3)^2(\psi')^2}{2\varepsilon^2 \alpha^2(n-1)(1+\phi)^2} \\
\quad \quad - \frac{2(\psi')^2}{\varepsilon^2(1+\phi)^2} - \frac{C_1}{1+\phi} - K - \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 f^2}{(n-1)f_1^2}.
\]
Hence
\begin{equation}
0 \geq \frac{(1 - \alpha^2) \mu^2}{2(n-1)(\mu \sup f - f)^2} - \frac{[(2n-3)^2 + (4n-5)\alpha^2](\psi')^2}{2\varepsilon^2 \alpha^2(n-1)(1 + \phi)^2} + \frac{C_1}{1 + \phi} + K + \frac{n \mu \sup f}{\mu \sup f - f} - \frac{\eta_1 \mu \sup f}{(n-1)f_1^2}.
\end{equation}

Multiplying through by \((1 + \phi)^4 \frac{\mu \sup f}{(\mu \sup f - f)^2}\), (2.24) becomes
\begin{equation}
0 \geq \frac{1 - \alpha^2}{2(n-1)} G^2 - \frac{[(2n-3)^2 + (4n-5)\alpha^2](\psi')^2}{2\varepsilon^2 \alpha^2(n-1)} + (1 + \phi)C_1 + (1 + \phi)^2 K \frac{n \mu \sup f}{\mu \sup f - f}
+ \frac{\eta_1 \mu \sup f}{\mu \sup f - f} - \frac{\eta_1^2 \mu^2(1 + \phi)^4}{(n-1)(\mu \sup f - f)^2}.
\end{equation}

This implies Theorem 2.1.

3. A COUNTEREXAMPLE

In this section, we shall show that the "interior rolling \(\varepsilon\)-ball" condition is necessary for \(\eta_1\) being bounded away from zero. We consider the following well-known example of E. Calabi [1]. For the sake of completeness, we will describe the example here.

Example 3.1. Let \(\Omega \subset \mathbb{R}^2\) be a domain as in Figure 1. The rectangle connecting the two disks is to be thought of as having fixed length \(l\) and variable width \(w\).

Let \(f\) be a function which is equal to \(c\) on the right-hand disk, \(-c\) on the left-hand disk and changes linearly from \(c\) to \(-c\) across the rectangle. (\(c\) is chosen to that \(\int_{\Omega} f^2 = 1\). Then \(\int_{\Omega} f = 0\) and
\[
\|\nabla f\| = \begin{cases} 
0 & \text{on the disks}, \\
\frac{2c}{l} & \text{on the rectangle}.
\end{cases}
\]

It is clear that
\[
\eta_1 = \inf_{h \in H^2_0(\Omega) \text{ such that } \int_{\Omega} h^2 = 1} \frac{\int_{\Omega} |\nabla h|^2}{\int_{\Omega} h^2}
\leq \int_{\Omega} |\nabla f|^2
= \frac{2c}{l}wl \to 0, \quad \text{as } w \to 0.
\]
This example shows that in bounding \( \eta_1 \) from below, it is necessary to consider the "interior rolling \( \varepsilon \)-ball" condition.

4. Proof of Theorem 1.1

In this section, we let \( f \) be a function which satisfies (1.3) with \( \eta = \eta_1 \); i.e., \( f \) is an eigenfunction of (1.3) with eigenvalue \( \eta_1 \). We denote \( N \) the nodal set of \( f \); i.e., \( N = \{ x \in M | f(x) = 0 \} \).

**Proof.** From Theorem 2.1, we know that

\[
\frac{\| \nabla f \|}{\mu \sup f - f} \leq \max \left\{ \frac{\sqrt{4(n-1)}}{\sqrt{1-\alpha^2}} \left[ C + \frac{(H+1)^2 \eta_1 \mu}{\mu - 1} \right]^{1/2}, \frac{\sqrt{S(H+1)}}{\sqrt{1-\alpha^2}} \left( \frac{\eta_1^{1/2}}{(\mu - 1)^{1/2}} \right) \right\}
\]

for any constant \( \mu > 1 \).

However, since \( f \) satisfies

\[
\int_M f = 0
\]

and

\[
f \not\equiv 0,
\]

this implies that the nodal set \( N \) of \( f \) divides \( M \) into two parts. If \( x \in M \) is a point where \( f \) achieves its supremum and \( \gamma \) is a shortest geodesic joining \( x \) and \( N \), then \( \gamma \) has length at most diameter of \( M \). Integrating (4.1) along \( \gamma \), we have

\[
\log \frac{\mu}{\mu - 1} \leq \int_\gamma \frac{\| \nabla f \|}{\mu \sup f - f} \leq 4\sqrt{(n-1)} \left[ C + \frac{(H+1)^2 \mu \eta_1}{\mu - 1} \right]^{1/2} d.
\]
Hence

\[
\frac{\mu - 1}{(H + 1)^2 \mu} \left[ \frac{1 - \alpha^2}{4(n - 1)d^2} \left( \log \frac{\mu}{\mu - 1} \right)^2 - C \right] \leq \eta_1.
\]

It is clear that the left-hand side can be made to be positive by choosing \( \mu \) close enough to 1. The theorem is then proved by maximizing (4.5) with

\[
\frac{\mu}{\mu - 1} = \exp \left[ 1 + \left( 1 + \frac{4(n - 1)d^2}{1 - \alpha^2} C \right)^{\frac{1}{2}} \right].
\]

This proves Theorem 1.1.

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References


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