ON DUAL BANACH ALGEBRAS

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Abstract. Let $A$ be a semisimple Banach algebra with $\| \cdot \|$, which is a dense subalgebra of a semisimple Banach algebra $B$ with $\| \cdot \|$ such that $\| \cdot \|$ majorizes $\| \cdot \|$ on $A$. The purpose of this paper is to investigate the dual property between the algebras $A$ and $B$. Some well-known results follow from this paper.

1. Introduction

Let $A$ be a semisimple Banach algebra with norm $\| \cdot \|$ which is a dense subalgebra of a semisimple Banach algebra $B$ with $\| \cdot \|$ such that $\| \cdot \|$ majorizes $\| \cdot \|$ on $A$. The purpose of this paper is to investigate the dual property between the algebras $A$ and $B$.

It is shown that if $A$ is a dual algebra, then $B$ is a dual algebra if and only if, $R = cl_B(R \cap A)$, for any proper closed right (left) ideal $R$ of $B$. On the other hand, if $B$ is a dual algebra, then $A$ is a dual algebra if and only, for any proper closed right (left) ideal $N$ of $A$, $N = cl_B(N) \cap A$ and for any proper closed right (left) ideal $R$ of $B$, $R = cl_B(R \cap A)$. If $A$ is a two-sided ideal of $B$ and $B$ has a bounded right approximate identity and a bounded left approximate identity, then we show that $A$ is a dual algebra if and only if $B$ is a dual algebra and $x \in cl_A(xA) \cap cl_A(Ax)$ for all $x$ in $A$. Some well-known results follow from our results.

2. Notation and preliminaries

Definitions not explicitly given are taken from Rickart [5].

Let $A$ be a Banach algebra. For any subset $E$ of $A$, let $cl_A(E)$ denote the closure of $E$ in $A$ and $\ell_A(E)$ (resp. $r_A(E)$) the left (resp. right) annihilator of $E$ in $A$. Then $A$ is called an annihilator algebra if $\ell_A(A) = r_A(A) = (0)$ and if for every proper closed right ideal $I$ and every proper closed left ideal $J$ $\ell_A(I) \neq (0)$ and $r_A(J) \neq (0)$. If, in addition, $r_A(\ell_A(I)) = I$ and $\ell_A(r_A(J)) = J$, then $A$ is called a dual algebra.

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An idempotent $e$ in a Banach algebra $A$ is said to be minimal if $e A e$ is a division algebra. In case $A$ is semisimple, this is equivalent to saying that $A e$ (resp. $e A$) is a minimal left (resp. right) ideal of $A$.

We say that a Banach algebra $A$ has a right approximate identity if there exists a net $\{u_t\}$ in $A$ such that $x = \lim xu_t$ for all $x$ in $A$. $\{u_t\}$ is not necessarily bounded. Analogously we define a left approximate identity.

Notation. If $A$ is a Banach algebra which is a dense subalgebra of a Banach algebra $B$, then we write $\| \cdot \|$ for the norm on $A$ and $| \cdot |$ for the norm on $B$.

In this paper, all algebras and linear spaces under consideration are over the field $C$ of complex numbers.

3. Banach algebra which is a dense subalgebra in another Banach algebra

In this section, $A$ will be a semisimple Banach algebra which is a dense subalgebra of a semisimple Banach algebra $B$ such that $\| \cdot \|$ majorizes $| \cdot |$ on $A$.

Let $E$ be a subset of $A$. Then it is clear that $\ell_B(E) = \ell_B(cl_A(E)) = \ell_B(cl_B(E))$ and $r_B(E) = r_B(cl_A(E)) = r_B(cl_B(E))$.

Lemma 3.1. Let $A$ be a dual algebra. Then for any proper closed right ideal $J$ of $A$, we have

\[ J = cl_B(J) \cap A = r_B(\ell_B(J)) \cap A. \]

Proof. Since $\ell_A(J) \subset \ell_B(cl_B(J))$, $\ell_B(cl_B(J)) \neq (0)$ and so $cl_B(J) \neq B$. Hence $cl_B(J)$ is a proper closed right ideal of $B$ and so

\[ J \subset cl_B(J) \cap A \subset r_B(\ell_B(cl_B(J))) \cap A \]
\[ = r_B(\ell_B(J)) \cap A \subset r_B(\ell_A(J)) \cap A \]
\[ = r_A(\ell_A(J)) = J. \]

Therefore $J = cl_B(J) \cap A = r_B(\ell_B(J)) \cap A$.

Theorem 3.2. Let $A$ be a dual algebra. Then the following statements are equivalent:

1. $B$ is a dual algebra.
2. For any $x$ in $B$, $x \in cl_B(xB) \cap cl_B(Bx)$.
3. For any proper closed right (left) ideal $R$ of $B$, $R = cl_B(R \cap A)$.

Proof. By [11, p. 79, Theorem 3.2], $B$ is an annihilator algebra and $A$ and $B$ have the same socle $S$, which is dense in both $A$ and $B$.

1. $\Rightarrow$ 2. It follows from [5, p. 105, Corollary (2.8.3)].

2. $\Rightarrow$ 3. Assume (2). Let $R$ be a proper closed right ideal of $B$ and $x \in R$. Since the socle $S$ is dense in $B$, we have $x = \lim_n xy_n$ in $| \cdot |$ with $y_n$ in $S$. Since $xy_n \in S \subset A$, we have $xy_n \in R \cap A$. Hence $x \in cl_B(R \cap A)$ and so $R \subset cl_B(R \cap A)$. Therefore $R = cl_B(R \cap A)$. Similarly, we can show that $R = cl_B(R \cap A)$, if $R$ is a proper closed left ideal of $B$. Therefore (3) is true.

3. $\Rightarrow$ 1. Assume (3). Let $R$ be a proper closed right ideal of $B$. Then by [5, p. 98, Corollary (2.8.7)], $R$ is contained in a maximal modular right ideal $M$.
of \( B \). Therefore by [5, p. 97, Theorem (2.8.5)], \( r_B(\ell_B(R)) \subset r_B(\ell_B(M)) = M \neq B \). Hence \( r_B(\ell_B(R)) \) is a proper closed right ideal of \( B \). Let \( J = \text{cl}_{A}(R \cap A) \). Then \( \text{cl}_{B}(J) = \text{cl}_{B}(R \cap A) = R \). Since \( r_B(\ell_B(J)) = \text{cl}_{B}(r_B(\ell_B(J)) \cap A) \), it follows from Lemma 3.1 that

\[
r_B(\ell_B(R)) = r_B(\ell_B(J)) = \text{cl}_{B}(r_B(\ell_B(J)) \cap A) = \text{cl}_{B}(J) = R.
\]

Similarly, we can show that \( \ell_B(r_B(R)) = R \), if \( R \) is a closed left ideal of \( B \). Therefore \( B \) is a dual algebra. This completes the proof of the theorem.

**Corollary 3.3.** Suppose that \( B \) has a left approximate identity and a right approximate identity. Then if \( A \) is a dual algebra, so is \( B \).

**Proof.** For any \( x \) in \( B \), it is clear that \( x \in \text{cl}_{B}(xB) \cap \text{cl}_{B}(Bx) \). Hence by Theorem 3.2, \( B \) is a dual algebra.

**Remark 1.** Let \( A \) be an \( A^* \)-algebra which is a dense subalgebra of a \( B^* \)-algebra \( B \). It is well known that if \( A \) is a dual algebra, so is \( B \). This result also follows from Corollary 3.3, because \( B \) has a bounded approximate identity.

**Remark 2.** A Banach algebra with an unbounded left approximate identity and unbounded right approximate identity may not have a bounded approximate identity (see [2, p. 487, Example 4.2]). On the other hand, if \( B \) is a dual algebra, \( A \) may not be a dual algebra. In fact, \( A \) may not be an annihilator algebra (for example, see [9, p. 1033] and [10, p. 293]).

The following result is useful in the next section.

**Theorem 3.4.** Let \( B \) be a dual algebra. Then the following statements are equivalent:

1. \( A \) is a dual algebra.
2. For any proper closed right (left) ideal \( N \) of \( A \), \( N = \text{cl}_{B}(N) \cap A \) and for any proper closed right (left) ideal \( R \) of \( B \), \( R = \text{cl}_{B}(R \cap A) \).

**Proof.** (1) \( \Rightarrow \) (2). Assume that \( A \) is a dual algebra. Since \( B \) is a dual algebra, by Theorem 3.2, \( R = \text{cl}_{B}(R \cap A) \). Let \( N \) be a proper closed right ideal of \( A \) and \( x \in \text{cl}_{B}(N) \cap A \). Then there exists a sequence \( \{x_n\} \subset N \) such that \( x_n \to x \) in \( | \cdot | \). Hence for any minimal idempotent \( e \) of \( A \), we have \( x_n e \to xe \) in \( | \cdot | \). Since by [11, p. 78, Lemma 3.1], \( \| \cdot \| \) and \( | \cdot | \) are equivalent on \( Ae \), \( x_n e \to xe \) in \( \| \cdot \| \). Since \( x_n e \in N \), \( xe \in N \), and so \( xA \subset N \). Since \( e \) is arbitrary, it follows that \( xS_A \subset N \), where \( S_A \) is the socle of \( A \) and so \( \text{cl}_{A}(xA) \subset N \). Therefore, by [5, p. 97, Corollary (2.8.3)], \( x \in \text{cl}_{A}(xA) \subset N \). Hence it follows that \( \text{cl}_{B}(N) \cap A \subset N \) and \( N = \text{cl}_{B}(N) \cap A \). A similar statement is true for left ideals. Consequently, (2) is true.

(2) \( \Rightarrow \) (1). Suppose that (2) is true. Let \( N \) be a proper closed right ideal of \( A \). Since \( N = \text{cl}_{B}(N) \cap A \), \( \text{cl}_{B}(N) \) is a proper closed right ideal of \( B \). Since \( B \) is a dual algebra, \( \ell_B(N) = \ell_B(\text{cl}_{B}(N)) \neq (0) \). Since \( \ell_B(N) \) is a proper closed
left ideal of $B$, by (2), $\ell_B(N) = cl_B(\ell_B(N) \cap A) = cl_B(\ell_A(N))$. In particular, $\ell_A(N) \neq (0)$. Also we have

$$N = cl_B(N) \cap A = r_B(\ell_B(cl_B(N))) \cap A$$
$$= r_B(\ell_B(N)) \cap A = r_B(cl_B(\ell_A(N))) \cap A$$
$$= r_B(\ell_A(N)) \cap A = r_A(\ell_A(N)).$$

Similarly, we can show that $J = \ell_A(r_A(J))$ for any closed left ideal $J$ of $A$. Therefore $A$ is a dual algebra and this completes the proof of the theorem.

**Corollary 3.5.** Assume that, for any proper closed right (left) ideal $R$ of $B$, $R = cl_B(R \cap A)$. Then the following statements are equivalent:

1. $A$ is a dual algebra.
2. $B$ is a dual algebra and, for any proper closed right (left) ideal $N$ of $A$, $N = cl_B(N) \cap A$.

**Proof.** This follows from Theorems 3.2 and 3.4.

The following result is essentially contained in [6, p. 262, Theorem 4.2].

**Theorem 3.6.** Let $B$ be a dual algebra. Then the following statements are equivalent:

1. $A$ is a dual algebra.
2. $A$ and $B$ have the same socle $S$ that is dense in $A$.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $A$ is a dual algebra. Then by [11, p. 79, Theorem 3.2], $A$ and $B$ have the same socle that is dense in $A$.

(2) $\Rightarrow$ (1). This follows from [6, p. 262, Theorem 4.2].

4. **Banach algebra which is a dense two-sided ideal in another Banach algebra**

In this section, $A$ will be a semisimple Banach algebra which is a dense two-sided ideal of a semisimple Banach algebra $B$. Then $\| \cdot \|$ majorizes $| \cdot |$ on $A$, there exists a constant $M$ such that

$$\|ab\| \leq M\|a\| |b| \text{ and } \|ba\| \leq M\|a\| |b|,$$

for all $a$ in $A$ and $b$ in $B$, and $A$ and $B$ have the same socle (see [11, p. 78, Lemma 2.1] and [1, p. 3]). (In [1], a slip is made in not assuming that $A = B \cdot A$ in Proposition 3.3 and Theorems 3.4 and 4.2.)

**Theorem 4.1.** Suppose that $B$ has a bounded right (resp. left) approximate identity $\{u_t\}$. Then $A$ has a right (resp. left) approximate identity if and only if $x \in cl_A(xA)$ (resp. $x \in cl_A(Ax)$) for all $x$ in $A$.

**Proof.** If $A$ has a right approximate identity, then clearly, $x \in cl_A(xA)$ for all $x$ in $A$.

Conversely, suppose that $x \in cl_A(xA)$ for all $x$ in $A$. By [2, p. 486, Lemma 2.1], we can assume that $\{u_t\} \subset A$. We show that $\{u_t\}$ is a right approximate identity of $A$. Since $\{u_t\}$ is bounded in $B$, there exists a constant $K$ such that
$|u_t| \leq K$ for all $t$. Let $x \in A$. Since $x \in cl_A(xA)$, for given $\varepsilon > 0$, there exists $y \in A$, such that $\|x - xy\| < \varepsilon/3MK(\varepsilon/3)$. Since $\{u_t\}$ is a right approximate identity of $B$, there exists $t_0$ such that, for $t > t_0$, $|y - yu_t| < \varepsilon/3M\|x\|$. Therefore,

$$\|x - xu_t\| \leq \|x - xy\| + \|xy - xyu_t\| + \|xyu_t - xu_t\|$$

$$\leq \|x - xy\| + M\|x\| |y - yu_t| + M\|xy - x\| |u_t|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$  

Therefore $\{u_t\}$ is a right approximate identity of $A$. This completes the proof.

Remark. If $A$ has a bounded right (or left) approximate identity, then $A = B$. In fact, suppose that $\{u_t\}$ is a bounded right approximate identity of $A$ with $\|u_t\| \leq K$, where $K$ is a constant. Then, for each $x \in A$, we have

$$\|x\| \leq \|x - e_t x\| + \|e_t x\| \leq \|x - e_t x\| + M\|e_t\| \|x\|$$

$$\leq \|x - e_t x\| + MK|x|.$$  

Since $\|x - e_t x\| \to 0$, it follows that $\|x\| \leq MK|x|$. Therefore $\| \cdot \|$ and $| \cdot |$ are equivalent on $A$ and so $A = B$.

Theorem 4.2. Suppose that $B$ has a bounded right approximate identity and a bounded left approximate identity. Then the following conditions are equivalent:

1. $A$ is a dual algebra.
2. $B$ is a dual algebra and $x \in cl_A(xA) \cap cl_A(Ax)$ for all $x \in A$.

Proof. (1) $\Rightarrow$ (2). Assume that $A$ is a dual algebra. Then by [5, p. 105, Corollary (2.8.3)], $x \in cl_A(xA) \cap cl_A(Ax)$ for all $x \in A$. Since $B$ has a bounded right approximate identity and a bounded left approximate identity, by Theorem 3.2, $B$ is a dual algebra.

(2) $\Rightarrow$ (1). Assume that (2) is true. Let $\{u_t\}$ be a bounded right approximate identity of $B$. Then by Lemma 4.1, we can assume that $\{u_t\}$ is a right approximate identity of $A$. Let $R$ be a closed right ideal of $B$ and $y \in R$. Since $yu_t \in R \cap A$ and $yu_t \to y$ in $| \cdot |$, it follows that $y \in cl_B(R \cap A)$. Therefore $R \subseteq cl_B(R \cap A)$, and so $R = cl_B(R \cap A)$. Let $N$ be a closed right ideal of $A$ and $x \in cl_B(N) \cap A$. Write $x = \lim_n x_n$ in $| \cdot |$ with $x_n \in N$. Let $z \in A$. Since $x_n z \in N$ and $\|xz - x_n z\| \leq M\|x - x_n\| \|z\|$, it follows that $xz \in N$; in particular $xu_t \in N$ for all $t$. Since $xu_t \to x$ in $\| \cdot \|$, it follows that $x \in N$. Therefore $cl_B(N) \cap A \subseteq N$ and so $N = cl_B(N) \cap A$. A similar statement is true for left ideals. Therefore, by Theorem 3.4, $A$ is a dual algebra. This completes the proof of the theorem.

The following result was proved by Johnson and Lahr (see [3, p. 313, Theorem 2]).

Corollary 4.3. Let $A$ be an $A^*$-algebra that is a dense two-sided ideal of a $B^*$-algebra $B$. Then $A$ is a dual algebra if and only if $B$ is a dual algebra and $A^2$ is dense in $A$. 

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Proof. Suppose that $A$ is a dual algebra. By Theorem 4.2, $B$ is a dual algebra. Since the socle of $A$ is dense in $A$, $A^2$ is dense in $A$.

Conversely, suppose that $B$ is a dual algebra and $A^2$ is dense in $A$. Then by [3, p. 312, Theorem 1], $A$ has an approximate identity, and so $x \in cl_A(xA) \cap cl_A(Ax)$ for all $x$ in $A$. Therefore by Theorem 4.2, $A$ is a dual algebra.

References