A UNIQUENESS CONDITION FOR FINITE MEASURES

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Abstract. Let \( \mu \) and \( \mu' \) be two finite measures on the same measurable space which have the property: \( \mu(A) = \mu(B) \) implies that \( \mu'(A) = \mu'(B) \). If the range of \( \mu \) is an interval, then there is a constant \( \alpha \) such that \( \mu' = \alpha \mu \). This extends earlier results of Leth and Malitz on purely atomic measures.

1. Introduction

Recently Malitz [6] proved the following result.

Theorem 1. Let \( \langle a_n \rangle \) and \( \langle a'_n \rangle \) be two sequences of real numbers such that

(i) \( a_n \to 0 \) and \( a'_n \to 0 \),

(ii) \( 0 < a_{n+1} \leq a_n \) and \( 0 < a'_n \leq a'_{n+1} \) for all \( n \),

(iii) \( a_n \leq \sum_{j>n} a_j \) and \( a'_n \leq \sum_{j>n} a'_j \) for all \( n \),

(iv) \( \sum_{i \in I} a_i = \sum_{j \in J} a_j \) iff \( \sum_{i \in I} a'_i = \sum_{j \in J} a'_j \).

Then there is a constant \( \alpha \) such that \( a'_n = \alpha a_n \) for all \( n \).

This theorem is a strengthening of an earlier result of Leth [4]. In Leth's theorem (iv) is replaced by

\[
\sum_{i \in I} a_i \leq \sum_{j \in J} a_j \text{ iff } \sum_{i \in I} a'_i \leq \sum_{j \in J} a'_j .
\]

Theorem 1 can be interpreted as a result on purely atomic measures, and in fact its origins [2] are in the study of purely atomic measures. (See also [1] and [7] for related results on nonatomic measures.) The purpose of this paper is to extend this result to arbitrary finite measures. The main result is

Theorem 2. Let \( \mu \) and \( \mu' \) be two finite measures on the same measurable space which have the property:

\( (*) \quad \mu(A) = \mu(B) \) implies \( \mu'(A) = \mu'(B) \).

If the range of \( \mu \) is an interval, then there is a constant \( \alpha \) such that \( \mu' = \alpha \mu \).
The proof of this theorem uses Theorem 1 and will be given in §4. Note
that if we take \( \mu \) in Theorem 2 to be a purely atomic measure we obtain the
following corollary, which is analogous to Theorem 1.

**Corollary 1.** Let \( \langle a_n \rangle \) and \( \langle a'_n \rangle \) be two sequences of real numbers such that

(i') \( \sum a_n \) and \( \sum a'_n \) converge,

(ii') \( 0 < a_{n+1} \leq a_n \) and \( 0 \leq a'_n \) for all \( n \),

(iii') \( a_n \leq \sum_{j > n} a_j \) for all \( n \),

(iv') \( \sum_{i \in I} a_i = \sum_{j \in J} a_j \) implies that \( \sum_{i \in I} a'_i = \sum_{j \in J} a'_j \).

Then there is a constant \( \alpha \) such that \( a'_n = \alpha a_n \) for all \( n \).

Comparing Corollary 1 with Theorem 1, note that (i') is a stronger condition
than (i). The reason is, of course, that Theorem 2 applies only to finite measures.
However, (ii'), (iii'), and (iv') are weaker than (ii), (iii), and (iv) respectively.

To see that Corollary 1 follows from Theorem 2, note that (iii') is a necessary
and sufficient condition for the range of the purely atomic measure correspond-
ing to the sequence \( \langle a_n \rangle \) to be an interval. This is included in Proposition 1 of
the next section.

2. Purely atomic case

In this section \( \langle a_n \rangle \) will be a sequence of positive real numbers with \( \sum a_n = s \)
and the terms ordered so that \( 0 < a_n \leq a_{n+1} \) for all \( n \). We also let \( r_n = \sum_{k > n} a_k \)
and denote the complement of a set \( A \) by \( A^c \).

The following result can be found in [4] and [6].

**Proposition 1.** For every \( x \in [0, s] \) there is a set \( I_x \) such that \( x = \sum_{n \in I_x} a_n \) iff
\( a_n \leq r_n \) for all \( n \). In this case, if \( x \in (0, s] \), \( I_x \) can be chosen so that \( I_x \) is
infinite. Also, if \( x \in [0, s] \), \( I_x \) can be chosen so that \( I_x^c \) is infinite.

**Remark 1.** If \( a_n \leq r_n \) for all \( n \), then for each \( n \) there is a set \( J_n \) such that
\( a_n = \sum_{k \in J_n} a_k \), and it is easy to see that we can choose \( J_n \) so that \( \min J_n = n+1 \).

In what follows we will assume that \( a_n \leq r_n \) for all \( n \) and that \( \langle a'_n \rangle \) is another
sequence of real numbers for which \( a'_n \geq 0 \) for all \( n \) and \( \sum a'_n \) converges. Let
\( s' = \sum a'_n \) and \( r'_n = \sum_{i > n} a'_i \) for all \( n \). We shall also assume that \( \langle a_n \rangle \) and
\( \langle a'_n \rangle \) satisfy

\[(**): \sum_{i \in I} a_i = \sum_{j \in J} a_j \implies \sum_{i \in I} a'_i = \sum_{j \in J} a'_j.\]

**Remark 2.** Referring to Remark 1, we have from (**) \( a'_n = \sum_{k \in J_n} a'_k \) with
\( \min J_n = n+1 \). Hence we see that \( a'_{n+1} \leq a'_n \) and \( a'_n \leq r'_n \).

**Proposition 2.** If \( a'_m = 0 \) for some \( m \), then \( a'_n = 0 \) for all \( n \).

**Proof.** Assume that \( a'_m = 0 \). By Remark 2, \( a'_{m+1} \leq a'_m = 0 \). Hence \( a'_{m+1} = 0 \)
and by an easy induction \( a'_n = 0 \) for \( n > m \).
set $J_{m-1}$ such that $a'_{m-1} = \sum_{k \in J_{m-1}} a'_k$ with $\min J_{m-1} = m$. Hence we have $a'_{m-1} = 0$. Therefore, by backwards induction, $a_n = 0$ for $n < m$.

In light of Proposition 2 we shall assume $a'_n > 0$ for all $n$. (If $a'_n = 0$ for some $n$, take $\alpha = 0$ in Theorem 3.)

We now define a function $f: [0,s] \to [0,s']$ by $f(x) = \sum_{n \in I_x} a'_n$ where $I_x$ is as given in Proposition 1. Note that $f(x)$ is independent of the choice of $I_x$ by (**). Also note that $f(s - x) = s' - f(x)$ for all $x \in [0,s]$ since $I_{s-x}$ can be taken to be $I_x^c$. We now prove some results about this function $f$.

**Proposition 3.** $f$ is continuous on $[0,s]$.

*Proof.* Let $\varepsilon > 0$ be given and let $x \in [0,s)$. Then by Proposition 1 we can choose a set $I_x$ so that $I_x^c$ is infinite and $x = \sum_{n \in I_x} a_n$. Choose $N$ so that $\sum_{n \geq N} a'_n < \varepsilon/2$ and let $\delta = \sum_{n \in I_x} a_n$. Note that $\delta > 0$ since $I_x^c$ is infinite. Then, if $x < y < x + \delta$, there is a set $J \subseteq \{ n : n \geq N \}$ for which $y = \sum_{n \in I_x} a_n + \sum_{n \in J} a_n$. Letting $x_0 = \sum_{n < N} a_n$ we have

$$|f(y) - f(x)| \leq |f(y) - f(x_0)| + |f(x) - f(x_0)| = \sum_{n \in J} a'_n + \sum_{n \in I_x} a'_n < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

Therefore $f$ is right continuous at $x$. Since $f(s - x) = s' - f(x)$, $f$ is also left continuous at $x$ for $x \in (0,s]$. Hence $f$ is continuous on $[0,s]$.

**Proposition 4.** $f$ is strictly increasing on $[0,s]$.

*Proof.* Assume that $f$ is not strictly increasing on $[0,s]$. Then, since $f$ is continuous, $f$ has an interior local maximum, say at $z$. Now $I_z$ may be chosen so that $I_z^c$ is infinite and $z = \sum_{n \in I_z} a_n$. Let $(a_{n_k})$ be a subsequence of $(a_n)$ for which $n_k \in I_z^c$ for all $k$. Then $f(z + a_{n_k}) = f(z) + a'_n > f(z)$ for all $k$. Hence there are points arbitrarily close to $z$ with larger function values than at $z$. This contradicts the fact that $f$ has a local maximum at $z$.

The following is an immediate consequence of Proposition 4.

**Corollary 2.** $f$ is one-to-one. Hence $\sum_{i \in I} a'_n = \sum_{j \in J} a'_j$ implies that $\sum_{i \in I} a_n = \sum_{j \in J} a_j$.

**Theorem 3.** Let $\mu$ and $\mu'$ be two finite measures on the same measurable space for which $\mu(A) = \mu(B)$ implies that $\mu'(A) = \mu'(B)$. If the range of $\mu$ is an interval and $\mu$ is purely atomic, then $\mu' = \alpha \mu$ for some constant $\alpha$.

*Proof.* Since the range of $\mu$ is an interval, $\mu$ has a countable infinity of atoms. Denote the sequence of $\mu$ measures of the atoms by $(a_n)$ and without loss of generality assume that $a_{n+1} \leq a_n$ for all $n$. Denote by $(a'_n)$ the $\mu'$ measures
of the atoms of \( \mu \) and note that condition (**) is satisfied. If \( a'_m = 0 \) for some \( m \), \( a'_n = 0 \) for all \( n \) by Proposition 2 and we take \( \alpha = 0 \). Now assume that \( a'_n > 0 \) for all \( n \). Then by Remark 2, Corollary 2, and construction, all the conditions of Theorem 1 are satisfied. Hence there is a constant \( \alpha \) such that \( a'_n = \alpha a_n \) for all \( n \) and therefore \( \mu' = \alpha \mu \).

3. Nonatomic case

In this section we use the following two results. The first can be found in [5, p. 100].

**Proposition 5.** If \( \mu \) is a nonatomic measure on a set \( X \), then the range of \( \mu \) is \([0, \mu(X)]\).

**Proposition 6.** If \( f \) is a function defined on \([0, a]\) which is continuous at 0 and additive (i.e., \( x, y, x + y \in [0, a] \) implies \( f(x + y) = f(x) + f(y) \)), then there is a constant \( \alpha \) such that \( f(x) = \alpha x \) for all \( x \in [0, a] \).

**Proof.** This result is well known in the case where the domain of \( f \) is \([0, \infty)\) but appears to be less well known as stated. For this reason we will give a sketch of the proof. First note that by replacing \( f(x) \) by \( f(x/a) \) we may assume that \( a = 1 \). Next by an easy induction one can show that if \( 0 \leq x \leq 1/n \), then \( f(nx) = nf(x) \). In particular \( f(1) = nf(1/n) \). Hence \( f(1/n) = f(1)m/n \) if \( 0 \leq m \leq n \). It is easy to show that additivity and continuity at 0 implies continuity on \([0, 1]\). Finally, continuity and \( f(x) = f(1)x \) for all rational \( x \) in \([0, 1]\) implies \( f(x) = \alpha x \) for all \( x \in [0, 1] \) with \( \alpha = f(1) \).

We shall use the following result to prove the main theorem of this section.

**Lemma 1.** Assume that \( \mu \) and \( \mu' \) are finite measures on the measurable space \((\Omega, \mathcal{B})\) which satisfy (*) \( \mu(A) = \mu(B) \) implies \( \mu'(\Omega) \neq 0 \). Then for any \( \varepsilon > 0 \) there is a set \( A \in \mathcal{B} \) such that \( 0 < \mu'(A) < \varepsilon \).

**Proof.** Given \( \varepsilon > 0 \) choose \( N \) such that \( \mu'(\Omega) < N \varepsilon \). Using Proposition 5 we can find disjoint sets \( A_1, A_2, \ldots, A_N \in \mathcal{B} \) such that \( \Omega = \bigcup_{n=1}^N A_n \) and \( \mu(A_1) = \mu(A_2) = \cdots = \mu(A_N) \). Hence by (\*),

\[
\mu'(A_1) = \mu'(A_2) = \cdots = \mu'(A_N) = \mu'(\Omega)/N < \varepsilon.
\]

Taking \( A = A_1 \), the lemma is proved.

**Theorem 4.** Let \( \mu \) and \( \mu' \) be two finite measures on the measurable space \((\Omega, \mathcal{B})\) which satisfy

\[
(*) \quad \mu(A) = \mu(B) \implies \mu'(A) = \mu'(B).
\]

If \( \mu \) is nonatomic, then \( \mu' = \alpha \mu \) for some constant \( \alpha \).

**Proof.** If \( \mu'(\Omega) = 0 \), take \( \alpha = 0 \). Now assume \( \mu'(\Omega) > 0 \). Define a function \( f: [0, \mu(\Omega)] \to [0, \mu'(\Omega)] \) as follows. If \( x \in [0, \mu(\Omega)] \) there is, by Proposition
5, a set $A_x \in \mathcal{B}$ such that $\mu(A_x) = x$. Let $f(x) = \mu'(A_x)$. By (*), $f(x)$ is independent of the choice of $A_x$.

We will first show that $f$ is continuous at 0. Let $\varepsilon > 0$ be given. By Lemma 1, there is a set $A \in \mathcal{B}$ such that $0 < \mu'(A) < \varepsilon$. Let $\delta = \mu(A)$. Note that $\delta > 0$ since if $\mu(A) = 0 = \mu(\phi)$, then $\mu'(A) = \mu'(\phi) = 0$. Now if $0 \leq x < \delta$, then by Proposition 5 there is a set $A_x \subseteq A$ such that $x = \mu(A_x)$. Hence

$$|f(x) - f(0)| = f(x) = \mu'(A_x) \leq \mu'(A) < \varepsilon$$

and we see that $f$ is continuous at 0.

Secondly we show that $f$ is additive. Assume that $x, y, x + y \in [0, \mu(\Omega)]$. Then by Proposition 5 there is a set $A_x \in \mathcal{B}$ such that $x = \mu(A_x)$ and a set $A_y \in \mathcal{B}$ such that $A_y \subseteq A_x$ and $y = \mu(A_y)$. Then

$$f(x + y) = f(\mu(A_x \cup A_y))$$

$$= \mu'(A_x \cup A_y)$$

$$= \mu'(A_x) + \mu'(A_y)$$

$$= f(x) + f(y).$$

Now by Proposition 6, there is a constant $\alpha$ such that $f(x) = \alpha x$ for all $x \in [0, \mu(\Omega)]$. Hence $\mu' = \alpha \mu$.

4. General case

We now give the proof of Theorem 2. In Theorem 3 we addressed the case where $\mu$ is purely atomic and in Theorem 4 the case where $\mu$ is nonatomic. We now consider the mixed case. Again let $(\Omega, \mathcal{B})$ be the measurable space on which $\mu$ and $\mu'$ are defined. Let $\Omega = C \cup D$, where $D$ is the union of all the atoms of $\mu$. We now consider the case where $\mu(C) \neq 0$ and $\mu(D) \neq 0$. Let $\mathcal{B}_C = \{A \cap C: A \in \mathcal{B}\}$ and let $\mu_C$ and $\mu'_C$ be the restrictions of $\mu$ and $\mu'$ respectively to $\mathcal{B}_C$. Now by Theorem 4 there is a constant $\alpha$ such that $\mu'_C = \alpha \mu_C$. Suppose that $E \in \mathcal{B}$ and $\mu(E) \leq \mu(C)$. Then there is a set $B \in \mathcal{B}_C$ for which $\mu(E) = \mu(B)$. Hence by (*), $\mu'(E) = \mu'(B) = \alpha \mu(B) = \alpha \mu(E)$. Now let $A_1, A_2, \ldots, A_m$ be those atoms of $\mu$ for which $\mu(A_i) > \mu(C)$ ordered such that $\mu(A_1) \leq \mu(A_2) \leq \cdots \leq \mu(A_m)$. Let $D' = D \sim \bigcup_{i=1}^{m} A_i$. By what we have just shown, for all measurable subsets of $C \cup D'$, $\mu'(E) = \alpha \mu(E)$. Now we show that there must be a measurable subset $E_1$ of $C \cup D'$ such that $\mu(E_1) = \mu(A_1)$. From Proposition 5, for any $x \in [0, \mu(C)]$ there is a subset $E$ of $C$ with $\mu(E) = x$. Now since all atoms of $D'$ have $\mu$ measure less than or equal to $\mu(C)$, it follows that for any $x \in [0, \mu(C \cup D')]$ there is a subset $E$ of $C \cup D'$ such that $\mu(E) = x$. Now $\mu(A_i) \leq \mu(C \cup D')$, since otherwise the interval $(\mu(C \cup D'), \mu(A_i))$ would be disjoint from the range of $\mu$. Consequently, there is indeed a subset $E_1$ of $C \cup D'$ for which $\mu(E_1) = \mu(A_1)$. Hence by (*), $\mu'(E_1) = \mu'(A_1)$. Therefore, $\mu'(A_1) = \mu'(E_1) = \alpha \mu(E_1) = \alpha \mu(A_1)$. Similarly there is a measurable subset $B_2$ of $C \cup D' \cup A_1$ such that $\mu(E_2) = \mu(A_2)$, and
as before, $\mu'(A_2) = \alpha \mu(A_2)$. Continuing, we obtain, by finite induction that $\mu'(A_m) = \alpha \mu(A_m)$. Hence $\mu'(E) = \alpha \mu(E)$ for all $E \in \mathcal{B}$.

5. Questions on possible extensions

It is natural to ask whether one can extend Theorem 2 to $\sigma$-finite measures. At present the author is unable to do this for arbitrary $\sigma$-finite measures, but there are two special cases which deserve mention.

First, if $\mu$ is a $\sigma$-finite measure with a nontrivial nonatomic part, then the result of Theorem 2 holds. This can be shown with slight modifications to the proof of Theorem 2 given in the last section.

Second, if $\mu$ is a purely atomic $\sigma$-finite measure for which the $\mu$ measures of the atoms decrease to 0, then the result of Theorem 2 also holds. This can be shown using results similar to those of §2 and using Theorem 1. Recall that Theorem 1 also holds for divergent series.

It is also natural to ask whether one can relax the requirement in Theorem 2 that the range of $\mu$ be an interval. Guthrie and the author have recently shown [3] that the range of any finite measure is always one of the following:

(1) a finite set,
(2) a finite union of closed intervals,
(3) homeomorphic to the Cantor set, or
(4) homeomorphic to a set described in detail in [3].

(For our purposes it is sufficient to know that this set and its complement both contain infinitely many intervals.)

If the range of $\mu$ is a finite set it is not difficult to see that the conclusion of Theorem 2 does not hold unless the range of $\mu$ is $\{ka: k = 0, 1, \ldots, n\}$ for $n = 0, 1, 2, \ldots$.

The following example shows that the conclusion of Theorem 2 does not hold if the range of $\mu$ is a union of two disjoint intervals. Similar examples can be constructed where the range of $\mu$ is a union of more than two disjoint intervals.

Example. Let $\mu$ and $\mu'$ be the purely atomic measures determined by the sequences $\langle a_n \rangle$ and $\langle a'_n \rangle$ respectively, defined by $a_n = a'_n = 1/2^n$ for $n \geq 1$ and $a_0 = a$ and $a'_0 = b$ with $a > b > 1$. Then the range of $\mu$ is easily seen to be $[0, 1] \cup [a, a + 1]$ and the range of $\mu'$ is $[0, 1] \cup [b, b + 1]$. Hence $\mu'$ is not a multiple of $\mu$. It is easy to check that condition $(\ast)$ is satisfied.

Leth [4] has given examples where the range of $\mu$ is homeomorphic to the Cantor set and the conclusion of Theorem 2 does not hold, but has also given an example where the conclusion does hold.

At present very little is known about the case where the range is a set of the fourth type.
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