GLOBAL $C^r$ STRUCTURAL STABILITY OF VECTOR FIELDS ON OPEN SURFACES WITH FINITE GENUS

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Abstract. A vector field $X$ on the open manifold $M$ is globally $C^r$ structurally stable if $X$ has a neighborhood $U$ in the Whitney $C^r$ topology such that the trajectories of every vector field $Y \in U$ can be mapped onto trajectories of $X$ by a homeomorphism $h: M \to M$ which is in a preassigned compact-open neighborhood of the identity. In [2] it was proved the theorem formulating the sufficient conditions for global $C^r$ ($r \geq 1$) structural stability of vector fields on open surfaces $(\dim M = 2)$. These conditions are also necessary for global $C^r$ structural stability on the plane if $r \geq 1$ (see [2]) and for $r = 1$ on any open surface of finite genus [1]. Here we will generalize it for $C^r$ ($r \geq 1$) vector fields defined on open orientable surface with finite genus and countable space of ends $E$.

Definitions and statement of the result

Let $M$ be an open orientable surface with finite genus and countable space of ends $E$ (also called the ideal boundary of $M$). A boundary component of surface $M$ is a nested sequence $P_1 \supset P_2 \supset \cdots$ of connected unbounded regions in $M$ such that:

(a) the boundary of $P_n$ in $M$ is compact for all $n$,
(b) for any boundary subset $K$ of $M$, $P_n \cap K = \emptyset$ for $n$ sufficiently large.

We say that two boundary components $P_1 \supset P_2 \supset \cdots$ and $P'_1 \supset P'_2 \supset \cdots$ are equivalent if for any $n$ there is a corresponding integer $m$ such that $P_m \subset P'_n$ and vice versa. Let $P^*$ denote the equivalence class of boundary components containing $P_1 \supset P_2 \supset \cdots$. The ideal boundary $E$ of a surface $M$ is the topological space having the equivalence classes of boundary components of $M$ as elements and endowed with such a topology that $E$ with it is homeomorphic to a subset of a Cantor set.

By $H^r(M)$ we denote the space of complete $C^r$ vector fields on $M$ with the $C^r$ Whitney topology ($r \geq 1$). $X$, $Y$ denote elements of $H^r(M)$, $\phi_X$ denotes the flow induced by $X$. For $x \in M$, $O_X(x)$ $(O^+_X(x), O^-_X(x))$ is the trajectory of $x$ (the positive semitrajectory, the negative semitrajectory) under $\phi_X$. By $O_X[x,y]$ we denote the closed $X$-trajectory segment from $x$ to $y$.
We distinguish three kinds of asymptotic behavior for each semitrajectory $O_X^±(x)$:

(a) $O_X^±(x)$ is bounded if it is contained in some compact set $K \subset M$,

(b) $O_X^±(x)$ escapes to infinity if for each compact set $K \subset M$ there is a point $y \in O_X^±(x)$ for which $O_X^±(y) \cap K = \emptyset$,

(c) $O_X^±(x)$ oscillates if it is neither bounded nor escapes to infinity.

These kinds of behavior for $O_X^+(x)$ (resp. $O_X^-(x)$) also can be described in terms of the $\omega$-limit (resp. $\alpha$-limit) set of $x \in M$ under $\phi_X$:

$$\omega(O_X^±(x)) = \{ y \in M : \exists t_n \to +\infty \text{ such that } \phi_X(x, t_n) \to y \},$$

$$\alpha(O_X^±(x)) = \{ y \in M : \exists t_n \to -\infty \text{ such that } \phi_X(x, t_n) \to y \}.$$

It is easy to see that $O_X^+(x)$

(a) is bounded iff $\omega(O_X^+(x))$ is compact (and nonempty),

(b) escapes to infinity iff $\omega(O_X^+(x)) = \emptyset$,

(c) oscillates iff $\omega(O_X^-(x))$ is a noncompact subset of $M$.

We extend the definition of $\omega$-limit (resp. $\alpha$-limit) set to $\omega^*$-limit (resp. $\alpha^*$-limit) set:

$$\omega^*(O_X^+(x)) = \{ y \in M \cup E : \exists t_n \to +\infty \text{ such that } \phi_X(x, t_n) \to y \},$$

$$\alpha^*(O_X^+(x)) = \{ y \in M \cup E : \exists t_n \to -\infty \text{ such that } \phi_X(x, t_n) \to y \}.$$

Thus

(a) $O_X^+(x)$ escapes to infinity iff there is $P^* \in E$ such that $\omega^*(O_X^+(x)) = P^*$,

(b) $O_X^+(x)$ oscillates iff $\omega(O_X^+(x)) \neq \emptyset$ and there is $P^* \in E$ such that $P^* \in \omega^*(O_X^+(x))$.

Let $\text{Per}(X), \Omega(X)$ denote, respectively, the set of periodic points, the set of nonwandering points of $X$, i.e.:

$$\text{Per}(X) = \{ x \in M : \phi_X(x, t) = x \text{ for some } t > 0 \},$$

$$\Omega(X) = \{ x \in M : \exists x_n \to x, t_n \to +\infty \text{ such that } \phi_X(x_n, t_n) \to x \}.$$

We define the first positive (resp. negative) prolongation limit set $x \in M$ by:

$$J_X^±(x) = \{ y \in M : \exists x_n \to x, t_n \to \pm\infty \text{ such that } \phi_X(x_n, t_n) \to y \}.$$

In general $\text{Per}(X) \subset \Omega(X)$, $\omega(O_X^+(x)) \subset J_X^+(x)$, $\alpha(O_X^-(x)) \subset J_X^-(x)$ and $\Omega(X) = \{ x \in M : x \in J_X^±(x) \}$.

Modifying the definition of Nemytskii–Stepanov [4], we say two unbounded semitrajectories $O_X^+(x)$ and $O_X^-(x)$ form a saddle at infinity if $y \in J_X^+(x)$ and $O_X^+(x), O_X^-(y)$ escape to infinity (i.e. there are points $P^*, Q^* \in E$ such that $\omega^*(O_X^+(x)) = P^*$, $\alpha^*(O_X^-(y)) = Q^*$). In this case, we call $O_X^+(x)$ (resp. $O_X^-(y)$) the stable (resp. unstable) separatrix of the saddle at infinity.

By $W_X^+$ (resp. $W_X^-$) we denote the union of all stable (resp. unstable) separatrices of fixed saddles and saddles at infinity. Each set is $\phi_X$ invariant; it
may consist of finitely or infinitely many distinct trajectories. In either case, it is not generally closed, since a fixed saddle belongs to the closure of its separatrices.

By a transverse section $S$ to a vector field $X \in H^r(M)$ we mean an embedded interval. A flowbox for $X \in H^r(M)$ is a closed quadrilateral $F \subset M$ containing no critical points of $X$, with two opposite edges $S^\pm$ transverse to $X$ and the other two edges $X$-trajectory segments, each joining an endpoint of $S^+$ to an endpoint of $S^-$. We call $S^+$ the entrance set and $S^-$ the exit set of $F$.

Let $H^r_0(M)$ be the set of vector fields $X \in H^r(M)$ with hyperbolic critical points, $H^r_{K-S}(M)$ be the set of Kupka-Smale vector fields, i.e. $X \in H^r_{K-S}(M)$ if $X \in H^r(M)$ and satisfies:

(a) for each $x \in \text{Per}(X)$ the trajectory $O_X(x)$ is hyperbolic,
(b) there are no saddle connections between fixed saddles.

In [5] it was proved that $H^r_0(M)$ is open and dense in $H^r(M)$ while $H^r_{K-S}(M)$ is residual in $H^r(M)$.

Recall that a minimal set is a nonempty compact invariant set with no proper compact invariant subsets. Trivial minimal sets are trajectories in $\text{Per}(X)$.

With these definitions, we can formulate sufficient conditions for global structural stability proved in [2].

**Theorem A.** Let $M$ be an open surface. If $X \in H^r(M)$ ($r \geq 1$) is a vector field satisfying:

(a) every trajectory in $\text{Per}(X)$ is hyperbolic,
(b) $X$ has no nontrivial minimal sets and no oscillating trajectories,
(c) $\text{cl} W^+_X \cap \text{cl} W^-_X \subset \text{Per}(X)$

then $\Omega(X) = \text{Per}(X)$ and $X$ is globally $C^r$ structurally stable.

**Theorem B.** Let $X \in H^r(M)$ be globally $C^r$ structurally stable vector field. Then conditions of Theorem A (a)-(c) hold if:

(a) $M = \mathbb{R}^2$ and $r \geq 1$,
(b) $M$ is any open surface of finite genus and $r = 1$.

Part (a) is proved in [2], part (b) in [1].

In the next section, we will prove the following:

**Theorem.** Let $M$ be an open orientable surface with finite genus and countable space of ends. Then conditions of Theorem A (a)-(c) are necessary for global $C^r$ ($r \geq 1$) structural stability of vector fields on $M$.

**Proof of the theorem.** We start with the theorem proved in [2].

**Proposition 1.** Let $X \in H^r(M)$ be globally $C^r$ structurally stable vector field, $r \geq 1$. Then every trajectory in $\text{Per}(X)$ is hyperbolic.

**Corollary 1.** If $X \in H^r(M)$ is globally $C^r$ ($r \geq 1$) structurally stable vector field then $X \in H^r_{K-S}(M)$.

**Proof.** $X$ is topologically conjugated to some vector field $Y \in H^r_{K-S}(M)$ since $H^r_{K-S}(M)$ is a residual subset of $H^r(M)$. By Proposition 1 every trajectory in
Per($X$) is hyperbolic. As $Y$ has no connection between fixed saddles and this property is preserved by conjugacy, it implies that $X \in H'_{K-S}(M)$.

Proposition 2. A globally $C^r$ ($r \geq 1$) structurally stable vector field has no nontrivial minimal sets.

Proof. Suppose that $K$ is a nontrivial minimal set of $X$. Let $U$ be a neighborhood of $X$ in $H'(M)$ such that for each $Y \in U$ there is a homeomorphism $h_Y$ of $M$ conjugating $X$ with $Y$. Let $Z \in U$ be $C^\infty$ vector field. Thus $h_Z(K)$ is a nontrivial minimal set of $Z$. We choose a neighborhood $U$ of $K$ with compact closure. Richards [6] proved that for any open surface $M$ with finite genus $g$ and space of ends $E$ there is a $C^\infty$ diffeomorphism $f: M \to N - E'$ where $N$ is a compact surface of genus $g$ and $E'$ is a closed, totally disconnected subset of $N$. Thus $Z' = Df(Z)$ is a vector field of class $C^\infty$ defined on $f(M)$ and $K' = f(h_Z(K))$ is contained in $U' = f(U) \subseteq N$. Applying a smooth partition of unity we may extend $Z'(a)$ to $C^\infty$ vector field $Z$, defined on $N$ such that $Z_1(y) = Z'(y)$ for $y \in K'$, $Z_1(y) = 0$ for $y \notin U'$. It implies that $K'$ is a nontrivial minimal set of $Z_1$. By Schwartz's theorem [7] any nontrivial minimal set of $C^2$ vector field on $N$ is the whole surface and $N$ is a two-dimensional torus which contradicts the property $K' \subseteq N$.

Proposition 3. Let $X \in H'(M)$ be globally $C^r$-structurally stable vector field, $r \geq 1$. Then $W^+_X \cap W^-_X = \emptyset$.

Proof. In [3] it was proved that $X$ has countably many stable and unstable separatrices of saddles at infinity. By Corollary 1 $X \in H'_{K-S}(M)$, so the union $W^\pm_X$ of all stable (unstable) separatrices of fixed saddles and saddles at infinity is also countable. Suppose that $W^+_X \cap W^-_X \neq \emptyset$. We choose flowboxes $F_1, F_2$ such that $\text{cl } F_1 \subset \text{int } F_2$, $W^+_X \cap W^-_X \cap S^+_1 \neq \emptyset$, where $S^+_1$ is the entrance set of $F_1$, $S^-_1$ is the exit set of $F_1$ and $\rho_H(F_1, F_2) < \varepsilon$ ($\rho$ is a Hausdorff metric). As $W^+_X, X^-_X$ are countable we may assume that edges of $F_2$ are not segments of trajectories in $W^+_X \cup W^-_X$. For every $O^+_X(x)$ in $W^+_X$ and crossing $S^-_2$ there is the last point of intersection. Analogously, there is the first point of intersection for every $O^-_X$ in $W^-_X$ and crossing $S^+_2$. We denote these sets, respectively, by $A$ and $B$. Let $V$ be the set of homeomorphisms of $M$ satisfying $\sup_{x \in F_i} \rho(h(x), \text{id}_M(x)) < \varepsilon$. Then $V$ is a neighborhood of $\text{id}_M$ in a compact-open topology. By assumption there is a neighborhood $U$ of $X$ in $H'(M)$ corresponding $V$ and homeomorphism $h_Y \in V$ conjugating $X$ with $Y$. Hence $W^+_Y \cap W^-_Y \cap F_2 \neq \emptyset$ for $Y \in U$. Let $Y(t) = X + \varepsilon tZ$, where $\varepsilon > 0$, $t \in [0, 1]$. $Z$ is $C^\infty$ vector field perpendicular to $X$ in $\text{int } F_2$ and $Z(x) = 0$ for $x \notin F_2$. For a sufficiently small $\varepsilon > 0$, $Y(t) \in U$ for all $0 \leq t \leq 1$. As $Y(t)(x) = X(x)$ for $x \notin F_2$ thus $O^+_Y(t)(a) = O^+_X(a)$ for $a \in A$, and $O^-_Y(t)(b) = O^-_X(b)$ for $b \in B$ which implies $O^+_Y(t)(a) \in W^+_Y(t)$, $O^-_Y(t)(b) \in W^-_Y(t)$. Moreover, other stable and unstable separatrices of saddles at
infinity are the same for $X$ and $Y(t)$, $t \in [0, 1]$. For every $b \in B$ there is a countable set $I_b \subset [0, 1]$ such that $O^+_Y(t)(B) \cap A \notin \emptyset$. Let $I = \bigcup_{b \in B} I_b$. Thus for $t \in [0, 1] - I$, $O^+_Y(t) \cap A = \emptyset$ and $W^+_Y(t) \cap W^-_Y(t) \cap F_2 = \emptyset$. So $X$ cannot be globally $C^r$ structurally stable since $X$ and $Y(t)$ are not conjugated on $F_1$ and $F_2$.

To prove the next proposition we need two lemmas.

**Lemma 1.** Let $S_1$, $S_2$ be compact transverse sections to $X \in H^r_K(M)$, $r \geq 1$, $P_X : D \to S_2$ be Poincaré map, where $D \subset S_1$. If $(a, b)$ is a component of $D$ then $O^+_X(a)$, $O^+_X(b) \in W^+_X$.

**Proof.** Let $x \in D$. Then $O^+_X(x) \cap S_2 \neq \emptyset$ and there is a neighborhood $U$ of $x$ in $S_1$ such that $O^+_X(y) \cap S_2 \neq \emptyset$ for $y \in U$. Thus $D$ is the countable union of open and connected subsets of $S_1$. Let $(a, b)$ or $(a, b]$ be such a component of $D$. We will show that $O^+_X(a) \in W^+_X$.

(i) Assume first that $\omega(O^+_X(a)) \neq \emptyset$. It is clear that if $\omega(O^+_X(a))$ contains a point $p \in \text{Per}(X)$ then $p$ is a fixed saddle. So either $\{p\} \subseteq \omega(O^+_X(a))$ or $\omega(O^+_X(a))$ contains stable and unstable separatrices of fixed saddle $p$. By $S'$ we denote a transverse section to $X$ at $y$ belonging to the unstable separatrix of $p$. For each $x \in (a, b)$ the set $A_x = O^+_X[x, P_X(x)] \cap S'$ is finite. Let $n_x = \text{card} \ A_x$. Since $P_X$ is a continuous map there is a number $n_0 \in \mathbb{N} \setminus \{0\}$ such that $n_x = n_0$ for $x \in (a, b)$. On the other hand the assumption $y \in \omega(O^+_X(a))$ implies that $O^+_X(a)$ crosses $S'$ infinitely many times. Then there exists $x_0 \in (a, b)$ sufficiently close to $a$ such that $n_{x_0} > n_0$. This proves that $\omega(O^+_X(a)) = \{p\}$.

(ii) Let $\omega(O^+_X(a)) = \emptyset$, i.e. $O^+_X(a)$ escapes to infinity. As $P_X$ is a homeomorphism $P_X(a, b) = (c, d) \subset S_2$ and $P_X^{-1}$ cannot be extended to $[c, d]$. Analogously like above, it is possible to prove that $\alpha$-limit set of $O^+_X(c)$ and $O^-_X(d)$ is either a fixed saddle or is empty. We have to show that $c \in J^+_X(a)$ or $d \in J^+_X(a)$. Let $x_n \in (a, b)$, $x_n \to a$. Then $P_X(x_n) = \phi_X(x_n, t_n) \in (c, d)$ and $P_X(x_n) \to c$ or $P_X(x_n) \to d$. Assume that $P_X(x_n) \to c$. If $\omega(O^+_X(a)) = \emptyset$ and $\omega(O^-_X(c)) = \emptyset$ then $t_n \to +\infty$ and $O^+_X(a)$, $O^-_X(c)$ form a saddle at infinity. If $\omega(O^+_X(a)) = \emptyset$ and $\omega(O^-_X(c))$ is a fixed saddle $p$, then stable separatrix $O^+_X(y)$ of $p$ is contained in the set of accumulation points of $O^+_X[x_n, P_X(x_n)]$. By (i) $\alpha(O^+_X(y))$ is empty or it is a fixed saddle. The last case is impossible since $X \in H^r_K(M)$. Thus $y \in J^+_X(a)$ and $O^+_X(a)$, $O^-_X(y)$ form a saddle at infinity. By the same arguments one can prove that $O^+_X(b) \in W^+_X$.

**Lemma 2.** Let $X \in H^r_K(M)$, $r \geq 1$, $O_X(a) \in W^+_X$, $O_X(b) \in W^-_X$ and $O_X(a) \subset \omega(O^+_X(b))$. Then for each neighborhood $U$ of $X$ in $H^r(M)$ there is a vector field $Y \in U$ such that $W^+_Y \cap W^-_Y \neq \emptyset$. 

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Proof. We choose a neighborhood $U$ of $X \in H'_{K}(M)$ and a flowbox $F$ such that $b \in \operatorname{int} F$. Let $S^\pm$ be, respectively, the entrance set and the exit set of $F$. $Z$ be a vector field perpendicular to $X$ at $\operatorname{int} F$ and $Z(x) = 0$ for $x \notin F$. Then for sufficiently small $\varepsilon > 0$ $Y(t) = X + \varepsilon t Z \in U$ for all $t \in [0, 1]$. By $u_1$, $u_2$ we denote the local coordinates defined on an open set $V \supset F$. We may assume that transverse sections $S^+$, $S^-$ are parallel to axis $u_2$. Let $a'$ be the first common point of $O^-_{X}(a)$ and $S^+$, $I$ be a closed neighborhood of $b'$ in $S^+$ such that $I \subset \operatorname{int} S^+$. We define $\eta = \inf_{x \in I} |u_2(x) - u_2(P_{Y(1)}(x))|$, where $P_{Y(1)}$ : $S^{+} \rightarrow S^{-}$ is a Poincaré map of vector field $Y(1)$. Since $I$ is compact $\eta > 0$. Let $a_0$ denote the last common point of $O_{X}(a)$ with $S^-$, $b_0$ the first common point of $O_{X}(b)$ with $S^+$. Then $O^{+}_{Y(t)}(a_0) = O^{+}_{X}(a_0)$, $O^{-}_{Y(t)}(b_0) = O^{-}_{X}(b_0)$ and $O^{+}_{Y(t)}(a_0) \in W^{+}_{Y(t)}$, $O^{-}_{Y(t)}(b_0) \in W^{-}_{Y(t)}$. Let $a_m$ be the $m$th intersection point of $O^{+}_{X}(a_0)$ with $S^-$, $b_n$ be the $n$th intersection point of $O^{+}_{X}(b_0)$ with $S^+$, $m \geq 0$, $n > 0$. Since $O_{X}(a) \subset \omega(O^{+}_{X}(b))$ then there are $a_m$, $b_n$ satisfying $|u_2(b_n) - u_2(a_m)| < \eta$.

(i) If $u_2(b_n) < u_2(a_m)$ we choose $Z$ directed as axis $u_2$. By $a_m(t)$, $b_n(t)$ we denote functions assigned, respectively, the $m$th intersection of $O_{Y(t)}(a)$ with $S^-$ and $n$th intersection of $O^{+}_{Y(t)}(b_0)$ with $S^+$ for $t \in [0, 1]$. It is clear that $a_m(t)$, $b_n(t)$ are continuous functions defined for sufficiently small $t$. Moreover, $u_2(a_m(t))$ decreases while $u_2(b_n(t))$ increases for $t \in [0, 1]$ since $M$ is an orientable surface. There are the following cases: either $a_m(t)$ and $b_n(t)$ are defined for all $t \in [0, 1]$ or at least one of them is not defined on the whole $[0, 1]$. In the first case there is $t_0 \in (0, 1)$ such that $u_2(b_n(t_0)) - u_2(a_m(t_0)) = 0$ and consequently $O^{+}_{Y(t_0)}(b_n(t_0)) = O^{-}_{Y(t_0)}(a_m(t_0)) \in W^{+}_{Y(t_0)} \cap W^{-}_{Y(t_0)}$. If $b_n(t)$ is not defined for all $t \in [0, 1]$ then $b_n(t_0)$ belongs to the boundary of some component of $D$, where $D \subset S^-$ and $P_{Y(t)} : D^+ \rightarrow S^+$ is a Poincaré map. By Lemma 1 $O^{+}_{Y(t_0)}(b_n(t_0)) \in W^{+}_{Y(t_0)}$, so $O^{-}_{Y(t_0)}(b_n(t_0)) \in W^{+}_{Y(t_0)} \cap W^{-}_{Y(t_0)}$. The proof is analogous if $a_m(t)$ is not defined for all $t \in [0, 1]$.

(ii) Suppose that $u_2(b_n) > u_2(a_m)$. Then we have to consider a vector field $Z$ directed opposite to axis $u_2$. Now $u_2(a_m(t))$ increases, $u_2(b_n(t))$ decreases for $t \in [0, 1]$ but the final arguments are the same as in (i).

The next lemma is proved in [3].

Lemma 3. Let $O^{+}_{X}(x)$ be an oscillating semitrajectory of vector field $X \in H'_{K-S}(M)$, $r \geq 1$. Then $\omega(O^{+}_{X}(x))$ contains a saddle at infinity.

Proposition 4. A globally $C^r$ ($r \geq 1$) structurally stable vector field $X \in H'_{K}(M)$ have no oscillating trajectories.

Proof. Suppose that $O^{+}_{X}(x)$ is an oscillating trajectory. We may assume that $O^{+}_{X}(x)$ oscillates. By Lemma 3 $\omega(O^{+}_{X}(x))$ contains a saddle at infinity, i.e. there are $O^{+}_{X}(a) \in W^{+}_{X}$, $O^{-}_{X}(b) \in W^{-}_{X}$ and $b \in J^{-}_{X}(a)$. Then either (i) $\omega(O^{+}_{X}(b)) = \emptyset$
or (ii) $\omega(O^+_X(b))$ is a compact set or (iii) $\omega(O^+_X(b))$ is noncompact subset of $M$. Since $O^+_X(b) \subset \omega(O^+_X(x)) \subset \Omega(X) = \{ y \in M : y \in J^+_X(y) \}$, $b \in J^+_X(b)$. If $\omega(O^+_X(b)) = \emptyset$ then $O^+_X(b)$ and $O^-_X(b)$ form a saddle at infinity and $O^+_X(b) \in W^+_X \cap W^-_X$. This contradicts Proposition 3. Assume that $\omega(O^+_X(b))$ is a nonempty compact set. Thus $\omega(O^+_X(b))$ contains a minimal set or it is such one. By Proposition 2 any minimal set is trivial so it is a fixed saddle. Suppose that $\omega(O^+_X(b))$ is a fixed saddle then $O^+_X(b) \in W^+_X \cap W^-_X$. In the other case $\omega(O^+_X(b))$ contains a stable separatrix $O^+_X(c)$ of a fixed saddle and $O^-_X(b)$, $O^+_X(c)$ satisfy assumptions of Lemma 2. Then for any neighborhood $U$ of $x$ in $H^r(M)$ there is a vector field $Y \in U$ satisfying $W^+_Y \cap W^-_Y \neq \emptyset$. Both cases are impossible by Proposition 3. Let $\omega(O^+_X(b))$ be noncompact subset of $M$, i.e. $O^+_X(b)$ oscillates. Lemma 3 implies that $\omega(O^+_X(b))$ contains a saddle at infinity $O^+_X(c)$ and $O^-_X(d)$. Applying again Lemma 2 we obtain a contradiction with Proposition 3. This proves Proposition 4.

**Proposition 5.** Let $X \in H^r(M)$ be globally $C^r$ structurally stable, $r \geq 1$. Then $\Omega(X) = \text{Per}(X)$.

**Proof.** By Corollary 1 and Propositions 3 and 4 $X \in H^r(M)$, $X$ has no oscillating trajectories and $W^+_X \cap W^-_X = \emptyset$. Suppose that there is $x \in \Omega(X) - \text{Per}(X)$. Then $\omega(O^+_X(x)) = \emptyset$ or $\omega(O^+_X(x))$ is a compact set. In the second case $\omega(O^+_X(x))$ is a fixed saddle and its unstable separatrix $O^+_X(a)$ escapes to infinity. Analogous possibilities are for $\alpha(O^-_X(x))$. Thus we have the following cases:

1. $\alpha(O^+_X(x)) = \emptyset$, $\omega(O^+_X(x)) = \emptyset$, $x \in J^+_X(x)$, so $O^+_X(x) \in W^+_X \cap W^-_X$.
2. $\alpha(O^-_X(x)) = \emptyset$, $\omega(O^+_X(x))$ is a fixed saddle and its unstable separatrix $O^+_X(a)$ escapes to infinity. Thus $a \in J^+_X(x)$, $O^+_X(x)$ and $O^+_X(a)$ form a saddle at infinity and $O^+_X(a) \in W^+_X \cap W^-_X$.
3. $\alpha(O^+_X(x)) = \emptyset$, $\omega(O^+_X(x))$ contains a fixed saddle, its stable separatrix $O_X(a)$ and unstable separatrix $O_X(b)$ such that $\alpha(O^+_X(a)) = \emptyset = \omega(O^+_X(b))$. Then $O^+_X(x)$ and $O^+_X(a)$ satisfy assumption of Lemma 2 and consequently $X$ is conjugated with a vector field $Y$ satisfying $W^+_Y \cap W^-_Y \neq \emptyset$. Thus also $W^+_X \cap W^-_X \neq \emptyset$ which contradicts our assumptions. The other three cases with $\omega(O^+_X(x)) = \emptyset$ are symmetric to (i)-(iii).

The next lemma is proved in [2].

**Lemma 4.** Let $F$ be a flowbox of $X \in H^r(M)$, $p \in \text{int} S^+$, $U$ be a neighborhood of $X$ in $H^r(M)$, $r \geq 1$. Then there exist a neighborhood $S^+_1$ of $p$ in $S^+$ and a flowbox $F_1 \subset F$ with entrance set $S^+_1$ and corresponding exit set $S^-_1$ such that for any pair of points $q^+ \in S^+_1$ there is a vector field $Y$ satisfying:

1. $Y(x) = X(x)$ for $x \notin F$,
2. $q^- \in O^+_Y(q^+)$ and $O_Y[q^+, q^-] \subset F$. 

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Proposition 6. For a globally $C^r$ ($r \geq 1$) structurally stable vector field $X \in H^r(M)$, $\operatorname{cl} W^+_X \cap \operatorname{cl} W^-_X \subset \operatorname{Per}(X)$.

Proof. Suppose that $x \in \operatorname{cl} W^+_X \cap \operatorname{cl} W^-_X - \operatorname{Per}(X)$. We choose a flowbox $F$ such that $x \in \operatorname{int} F$. Since $x \notin \operatorname{Per}(X)$ then by Proposition 5 $x \notin \Omega(X)$. Thus we may assume that $O^+_X(S^-) \cap F = \emptyset$, $O^-_X(S^+) \cap F = \emptyset$, where $S^+$, $S^-$ are, respectively, the entrance set and the exit set of $F$. Let $p = O^-_X(x) \cap S^+$, $q = O^+_X(x) \cap S^-$. Since $x \in \operatorname{cl} W^+_X \cap \operatorname{cl} W^-_X$ there are points $(p_n)$, $(q_n)$ such that $p_n \in S^+$, $q_n \in S^-$, $O^-_X(p_n) \in W^-_X$, $O^+_X(q_n) \in W^+_X$, $p_n \to p$, $q_n \to q$. Let $U$ be a neighborhood of $X$ in $H^r(M)$. By Lemma 4 there are transverse sections $S^+_1 \subset S^+$, $S^-_1 \subset S^-$, points $p_n \in S^+_1$, $q_n \in S^-_1$ and a vector field $Y \in U$ such that $q_n \in O_Y(p_n)$. Thus $O_Y(p_n) = O_Y(q_n) \in W^-_Y \cap W^+_Y$ and $X$ is not globally structurally stable by Proposition 3.

Propositions 1, 2, 4, and 6 imply the following:

Theorem. Let $X \in H^r(M)$ be globally $C^r$ ($r \geq 1$) structurally stable. Then:

(a) every trajectory in $\operatorname{Per}(X)$ is hyperbolic,
(b) $X$ has no nontrivial minimal sets and no oscillating trajectories,
(c) $\operatorname{cl} W^+_X \cap \operatorname{cl} W^-_X \subset \operatorname{Per}(X)$.

REFERENCES


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