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THE SEMIMARTINGALE STRUCTURE OF REFLECTING BROWNIAN MOTION

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ABSTRACT. We prove that reflecting Brownian motion on a bounded Lipschitz domain is a semimartingale. We also extend the well-known Skorokhod equation to this case.

In this note we study the semimartingale property and the Skorokhod equation of reflecting Brownian motion on a bounded Euclidean domain. A R^d-valued continuous stochastic process $X = \{X_t; t \geq 0\}$ is said to be a semimartingale if it can be decomposed into the form

$$X_t = X_0 + M_t + \tfrac{1}{2}N_t,$$

where M is a continuous martingale with zero initial value, and N (ignoring the factor $1/2$) is a process of bounded variation. Let $|N|$ be its total variation process, i.e.,

$$|N|_t = \sup \sum_{i=1}^{n-1} |N_{t_i} - N_{t_{i-1}}|.$$

Here the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_n = t$, and $|\cdot|$ denotes the Euclidean distance. We have the following expression

$$N_t = \int_0^t \nu_s d|N|_s$$

where ν is a process with length one, i.e., with probability one, $|\nu|_s = 1$ for $|N|$-almost all s.

The original Skorokhod equation refers to one-dimensional reflecting Brownian motion $X = |B|$ (B is a standard one-dimensional Brownian motion). It states that X is a semimartingale and $X_t = X_0 + W_t + \frac{1}{2}L_t$, where W is a standard Brownian motion and L is the local time of X at $x = 0$.

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Suppose that D is a bounded smooth domain in R^d. Let ν be the inward unit normal vector field on the boundary ∂D. Suppose that X is a reflecting Brownian motion on D. The multidimensional Skorokhod equation takes the form

$$X_t = X_0 + W_t + \tfrac{1}{2}\int_0^t \nu(X_s)\, dL_s, \tag{1}$$

where W is a standard d-dimensional Brownian motion and L is the boundary local time of X, i.e., the continuous additive functional of X associated with the surface measure of D. This form of the Skorokhod equation was first proved for convex domains in [T], then for C^1 domains by [LS] (see also [H]). In both cases, the stochastic Skorokhod equation is obtained by first solving a deterministic Skorokhod equation. As a matter of fact, (1) can be regarded as a stochastic differential equation with reflecting boundary conditions in two unknown processes X and L. The existence and uniqueness of the solution of the deterministic Skorokhod equation imply the existence and pathwise uniqueness of the solution of the stochastic Skorokhod equation.

A natural question at this point is how smooth the domain D has to be to insure that reflecting Brownian motion is a semimartingale. In this paper we will discuss bounded Lipschitz domains in any dimension. Our main result is that for these domains, reflecting Brownian motion is a semimartingale and the Skorokhod equation holds.

Let D be a bounded Lipschitz domain. First we must make sure that reflecting Brownian motion can be defined as a continuous $\overline{D}$-valued process. This fact follows from our previous work [BH]. For a discussion of reflecting Brownian motion on arbitrary domains, see [F1]. Further information on reflecting Brownian motion on Lipschitz and Hölder domains can be found in [BH].

Theorem 1. *Suppose that D is a bounded Lipschitz domain. Then reflecting Brownian motion X is a continuous $\overline{D}$-valued semimartingale, and the Skorokhod equation*

$$X_t = X_0 + W_t + \tfrac{1}{2}\int_0^t \nu(X_s)\, dL_s,$$

holds, where W is a standard d-dimensional Brownian motion, L is the boundary local time (continuous additive functional) associated with the surface measure σ on ∂D, and ν is the inward unit normal vector field on the boundary.

The inward pointing normal vector is only defined a.e. (with respect to surface measure). However, the continuous additive functional L is associated with σ and so does not charge the null set. Hence the integral in the statement of Theorem 1 is unambiguously defined.

We will give a proof of Theorem 1 based on our previous work on reflecting Brownian motion on Lipschitz domains. For general domains, the reflecting Brownian motion may not be a continuous process on the Euclidean closure of D. It is a continuous process on a special compactification of D, the so-called

Kuramochi compactification. In [BH], we have shown that if D is a bounded Lipschitz domain, then the Kuramochi compactification of D is the same as the Euclidean compactification. Thus, for such domains, the reflecting Brownian motion does live on the set $\overline{D}$. To show that it is actually a semimartingale, we use the theory of Dirichlet forms [F2].

Proof of Theorem 1. The Dirichlet form for reflecting Brownian motion is

$$\mathscr{E}(u,v) = \tfrac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)m(dx), \qquad D(\mathscr{E}) = H^1(D).$$

(m is the Lebesgue measure on D). In [BH] we proved that for D bounded and Lipschitz, this Dirichlet form is regular on $\overline{D}$, which means that the set $H^1(D)\cap C(\overline{D})$ is dense in both $H^1(D)$ and $C(\overline{D})$, each functional space being equipped with its usual norm. We can now make use of the theory of regular Dirichlet forms developed in [F2], especially Chapter 5.

Suppose that $f\in H^1(D)\cap C(\overline{D})$. According to Theorem 5.2.2 of [F2], the continuous additive functional $f(X_t)-f(X_0)$ can be decomposed as follows:

$$f(X_t)-f(X_0) = M_t^f + N_t^f, \tag{2}$$

where M^f is a martingale additive functional of finite energy and N^f is a continuous additive functional of zero energy. Since X has continuous sample paths and f is assumed to be continuous on $\overline{D}$, M^f is a continuous martingale whose quadratic variation process is

$$\langle M^f, M^f\rangle_t = \int_0^t |\nabla f|^2(X_x)\,ds. \tag{3}$$

(See Example 5.2.1 in [F2].) If we further assume that $f\in C^2(\overline{D})$, then by Theorem 5.3.2 of [F2], N^f is of bounded variation and its associated measure μ^f is uniquely characterized by the relation

$$\tfrac{1}{2}\int_D \nabla f(x)\cdot\nabla v(x)m(dx) = \int_{\overline{D}} \tilde{v}(x)\mu^f(dx), \qquad \forall v\in H^1(D).$$

($\tilde{v}$ is a quasi-continuous modification of v.) Since D is Lipschitz, we can use Green's identity in the above equation. This allows us to identify the associated measure of N^f, i.e.,

$$\mu^f(dx) = -\frac{1}{2}\Delta f(x)m(dx) + \frac{1}{2}\frac{\partial f}{\partial\nu}(x)\sigma(dx), \tag{4}$$

where σ is the surface measure of the boundary ∂D.

Now apply the above discussion to the coordinate functions $f_i(x) = x^i$. We have

$$X_t = X_0 + M_t + \frac{1}{2}N_t, \tag{5}$$

where $M(M^{f_1},\dots,M^{f_d})$, and $N = (N^{f_1},\dots,N^{f_d})$. It remains to show that M is a standard d-dimensional Brownian motion and $N_t = \int_0^t \nu(X_s)\,dL_s$.

To see that M is a Brownian motion, we use Lévy's criterion. Namely, we need to verify that

$$\langle M^{f_i}, M^{f_j}\rangle = \delta_{ij} t, \qquad i, j = 1, \dots, d.$$

This follows immediately from (3). Therefore M is a Brownian motion.

Let $\nu(x) = (\nu^1(x), \dots, \nu^d(x))$ be the components of the normal vector ν. From (4), the measure associated with the continuous additive functional N^{f_i} is $\nu^i(x)\sigma(dx)$. Let

$$L_t = \sum_{i=1}^{d} \int_0^t \nu^i(x)\, dN_s^{f_i}.$$

It follows that the measure associated with L is $\sum_{i=1}^d \nu^i(x)^2 \sigma(dx) = \sigma(dx)$. This shows that L is just the boundary local time with respect to the surface measure. Since the measure for N^{f_i} is $\nu^i(x)\sigma(dx)$, we have

$$N_t^{f_i} = \int_0^t \nu^i(X_s)\, dL_s, \qquad i = 1, \dots, d.$$

Hence we obtain

$$N_t = \int_0^t \nu(X_s)\, dL_s,$$

and the proof of the Skorokhod equation is complete. □

Remark. The tightness estimates of [BH, §2] allow us to construct reflecting Brownian motions on $\overline{D}$ when D is a Hölder domain in R^d, $d \geq 3$. Unless the Kuramochi compactification for such a domain D is equal to the Euclidean compactification, however, there will be more than one reflecting Brownian motion on $\overline{D}$, and the question of semimartingale representations loses some of its interest.

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