

## THE SEMIMARTINGALE STRUCTURE OF REFLECTING BROWNIAN MOTION

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(Communicated by George C. Papanicolaou)

**ABSTRACT.** We prove that reflecting Brownian motion on a bounded Lipschitz domain is a semimartingale. We also extend the well-known Skorokhod equation to this case.

In this note we study the semimartingale property and the Skorokhod equation of reflecting Brownian motion on a bounded Euclidean domain. A  $R^d$ -valued continuous stochastic process  $X = \{X_t; t \geq 0\}$  is said to be a semimartingale if it can be decomposed into the form

$$X_t = X_0 + M_t + \frac{1}{2}N_t,$$

where  $M$  is a continuous martingale with zero initial value, and  $N$  (ignoring the factor  $1/2$ ) is a process of bounded variation. Let  $|N|$  be its total variation process, i.e.,

$$|N|_t = \sup \sum_{i=1}^{n-1} |N_{t_i} - N_{t_{i-1}}|.$$

Here the supremum is taken over all finite partitions  $0 = t_0 < t_1 < \dots < t_n = t$ , and  $|\cdot|$  denotes the Euclidean distance. We have the following expression

$$N_t = \int_0^t \nu_s d|N|_s$$

where  $\nu$  is a process with length one, i.e., with probability one,  $|\nu|_s = 1$  for  $|N|$ -almost all  $s$ .

The original Skorokhod equation refers to one-dimensional reflecting Brownian motion  $X = |B|$  ( $B$  is a standard one-dimensional Brownian motion). It states that  $X$  is a semimartingale and  $X_t = X_0 + W_t + \frac{1}{2}L_t$ , where  $W$  is a standard Brownian motion and  $L$  is the local time of  $X$  at  $x = 0$ .

Received by the editors July 17, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 60J65; Secondary 60J35, 60J50.

*Key words and phrases.* reflecting Brownian motion, Lipschitz domain, Dirichlet form, Skorokhod equation.

The first author was partially supported by NSF Grant DMS 88-22053; the second author was partially supported by NSF Grant DMS 86-01977.

Suppose that  $D$  is a bounded smooth domain in  $R^d$ . Let  $\nu$  be the inward unit normal vector field on the boundary  $\partial D$ . Suppose that  $X$  is a reflecting Brownian motion on  $D$ . The multidimensional Skorokhod equation takes the form

$$(1) \quad X_t = X_0 + W_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s,$$

where  $W$  is a standard  $d$ -dimensional Brownian motion and  $L$  is the boundary local time of  $X$ , i.e., the continuous additive functional of  $X$  associated with the surface measure of  $D$ . This form of the Skorokhod equation was first proved for convex domains in [T], then for  $C^1$  domains by [LS] (see also [H]). In both cases, the stochastic Skorokhod equation is obtained by first solving a deterministic Skorokhod equation. As a matter of fact, (1) can be regarded as a stochastic differential equation with reflecting boundary conditions in two unknown processes  $X$  and  $L$ . The existence and uniqueness of the solution of the deterministic Skorokhod equation imply the existence and pathwise uniqueness of the solution of the stochastic Skorokhod equation.

A natural question at this point is how smooth the domain  $D$  has to be to insure that reflecting Brownian motion is a semimartingale. In this paper we will discuss bounded Lipschitz domains in any dimension. Our main result is that for these domains, reflecting Brownian motion is a semimartingale and the Skorokhod equation holds.

Let  $D$  be a bounded Lipschitz domain. First we must make sure that reflecting Brownian motion can be defined as a continuous  $\bar{D}$ -valued process. This fact follows from our previous work [BH]. For a discussion of reflecting Brownian motion on arbitrary domains, see [F1]. Further information on reflecting Brownian motion on Lipschitz and Hölder domains can be found in [BH].

**Theorem 1.** *Suppose that  $D$  is a bounded Lipschitz domain. Then reflecting Brownian motion  $X$  is a continuous  $\bar{D}$ -valued semimartingale, and the Skorokhod equation*

$$X_t = X_0 + W_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s,$$

*holds, where  $W$  is a standard  $d$ -dimensional Brownian motion,  $L$  is the boundary local time (continuous additive functional) associated with the surface measure  $\sigma$  on  $\partial D$ , and  $\nu$  is the inward unit normal vector field on the boundary.*

The inward pointing normal vector is only defined a.e. (with respect to surface measure). However, the continuous additive functional  $L$  is associated with  $\sigma$  and so does not charge the null set. Hence the integral in the statement of Theorem 1 is unambiguously defined.

We will give a proof of Theorem 1 based on our previous work on reflecting Brownian motion on Lipschitz domains. For general domains, the reflecting Brownian motion may not be a continuous process on the Euclidean closure of  $D$ . It is a continuous process on a special compactification of  $D$ , the so-called

Kuramochi compactification. In [BH], we have shown that if  $D$  is a bounded Lipschitz domain, then the Kuramochi compactification of  $D$  is the same as the Euclidean compactification. Thus, for such domains, the reflecting Brownian motion does live on the set  $\bar{D}$ . To show that it is actually a semimartingale, we use the theory of Dirichlet forms [F2].

*Proof of Theorem 1.* The Dirichlet form for reflecting Brownian motion is

$$\mathcal{E}(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) m(dx), \quad D(\mathcal{E}) = H^1(D).$$

( $m$  is the Lebesgue measure on  $D$ ). In [BH] we proved that for  $D$  bounded and Lipschitz, this Dirichlet form is regular on  $\bar{D}$ , which means that the set  $H^1(D) \cap C(\bar{D})$  is dense in both  $H^1(D)$  and  $C(\bar{D})$ , each functional space being equipped with its usual norm. We can now make use of the theory of regular Dirichlet forms developed in [F2], especially Chapter 5.

Suppose that  $f \in H^1(D) \cap C(\bar{D})$ . According to Theorem 5.2.2 of [F2], the continuous additive functional  $f(X_t) - f(X_0)$  can be decomposed as follows:

$$(2) \quad f(X_t) - f(X_0) = M_t^f + N_t^f,$$

where  $M^f$  is a martingale additive functional of finite energy and  $N^f$  is a continuous additive functional of zero energy. Since  $X$  has continuous sample paths and  $f$  is assumed to be continuous on  $\bar{D}$ ,  $M^f$  is a continuous martingale whose quadratic variation process is

$$(3) \quad \langle M^f, M^f \rangle_t = \int_0^t |\nabla f|^2(X_s) ds.$$

(See Example 5.2.1 in [F2].) If we further assume that  $f \in C^2(\bar{D})$ , then by Theorem 5.3.2 of [F2],  $N^f$  is of bounded variation and its associated measure  $\mu^f$  is uniquely characterized by the relation

$$\frac{1}{2} \int_D \nabla f(x) \cdot \nabla v(x) m(dx) = \int_{\bar{D}} \tilde{v}(x) \mu^f(dx), \quad \forall v \in H^1(D).$$

( $\tilde{v}$  is a quasi-continuous modification of  $v$ .) Since  $D$  is Lipschitz, we can use Green's identity in the above equation. This allows us to identify the associated measure of  $N^f$ , i.e.,

$$(4) \quad \mu^f(dx) = -\frac{1}{2} \Delta f(x) m(dx) + \frac{1}{2} \frac{\partial f}{\partial \nu}(x) \sigma(dx),$$

where  $\sigma$  is the surface measure of the boundary  $\partial D$ .

Now apply the above discussion to the coordinate functions  $f_i(x) = x^i$ . We have

$$(5) \quad X_t = X_0 + M_t + \frac{1}{2} N_t,$$

where  $M = (M^{f_1}, \dots, M^{f_d})$ , and  $N = (N^{f_1}, \dots, N^{f_d})$ . It remains to show that  $M$  is a standard  $d$ -dimensional Brownian motion and  $N_t = \int_0^t \nu(X_s) dL_s$ .

To see that  $M$  is a Brownian motion, we use Lévy's criterion. Namely, we need to verify that

$$\langle M^{f_i}, M^{f_j} \rangle = \delta_{ij}t, \quad i, j = 1, \dots, d.$$

This follows immediately from (3). Therefore  $M$  is a Brownian motion.

Let  $\nu(x) = (\nu^1(x), \dots, \nu^d(x))$  be the components of the normal vector  $\nu$ . From (4), the measure associated with the continuous additive functional  $N^{f_i}$  is  $\nu^i(x)\sigma(dx)$ . Let

$$L_t = \sum_{i=1}^d \int_0^t \nu^i(x) dN_s^{f_i}.$$

It follows that the measure associated with  $L$  is  $\sum_{i=1}^d \nu^i(x)^2 \sigma(dx) = \sigma(dx)$ . This shows that  $L$  is just the boundary local time with respect to the surface measure. Since the measure for  $N^{f_i}$  is  $\nu^i(x)\sigma(dx)$ , we have

$$N_t^{f_i} = \int_0^t \nu^i(X_s) dL_s, \quad i = 1, \dots, d.$$

Hence we obtain

$$N_t = \int_0^t \nu(X_s) dL_s,$$

and the proof of the Skorokhod equation is complete.  $\square$

*Remark.* The tightness estimates of [BH, §2] allow us to construct reflecting Brownian motions on  $\bar{D}$  when  $D$  is a Hölder domain in  $R^d$ ,  $d \geq 3$ . Unless the Kuramochi compactification for such a domain  $D$  is equal to the Euclidean compactification, however, there will be more than one reflecting Brownian motion on  $\bar{D}$ , and the question of semimartingale representations loses some of its interest.

## REFERENCES

- [BH] R. F. Bass and P. Hsu, *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains* (to appear in *Ann. Probability*).
- [F1] M. Fukushima, *A construction of reflecting barrier Brownian motions for bounded domains*, *Osaka J. Math.* **4** (1967), 183–215.
- [F2] —, *Dirichlet forms and Markov processes*, North-Holland, Amsterdam, 1980.
- [H] P. Hsu, *Reflecting Brownian motion, boundary local time, and the Neumann boundary value problem*, Ph.D. dissertation, Stanford, 1984.
- [LS] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, *Comm. Pure Appl. Math.* **37** (1984), 511–537.
- [T] H. Tanaka, *Stochastic differential equations with reflecting boundary condition in convex region*, *Hiroshima Math. J.* **9** (1967), 163–177.

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